Worksheet on free resolutions and Tor

Definition:

- A **free resolution** of a an R-module M is an exact sequence of the form
$$\cdots \to F_{i+1} \xrightarrow{d_i} F_i \xrightarrow{d_{i-1}} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_0} F_0 \to M \to 0,$$
where each module F_i is free (possibly zero).
- If $(R, \mathfrak{m})$ is local or graded, a free resolution as above is **minimal** if $\operatorname{im}(d_i) \subseteq F_i$ for all i.

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(1) Show that every module M over any ring has a free resolution.
(2) Show that if R is Noetherian, then every finitely generated module has a free resolution in which each F_i is finitely generated.
(3) Show that if R is Noetherian and local or graded, and M is finitely generated, then M has a minimal free resolution in which each F_i is finitely generated; in the graded case each F_i is graded.
(4) Find a minimal free resolution of $K[x]/(x)$ over $K[x]$, and a free resolution that is not minimal.
(5) Find a minimal free resolution of $K[x,y]/(x,y)$ over $K[x,y]$, and a free resolution that is not minimal.
(6) Find a minimal free resolution of $K[x,y]/(x^2, xy)$ over $K[x,y]$.
(7) Find a minimal free resolution of $R/(x,y)$ over $R = K[x,y]/(xy)$.

Definition: If M is an R-module, the **Tor functors** are defined as follows: first, take a free resolution of M as above.

- For a module N, with a free resolution of M as above, we have $\operatorname{Tor}_i^R(M,N) := \dfrac{\ker(d_{i-1} \otimes N)}{\operatorname{im}(d_i \otimes N)}$.
- For a homomorphism $N \xrightarrow{\alpha} N'$, we have
$$(F_i \otimes \alpha)(\ker(d_{i-1} \otimes N)) \subseteq \ker(d_{i-1} \otimes N') \text{ and } (F_i \otimes \alpha)(\operatorname{im}(d_i \otimes N)) \subseteq \operatorname{im}(d_i \otimes N').$$
Thus, the map $F_i \otimes \alpha$ induces a map $\operatorname{Tor}_i^R(M,\alpha) : \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N')$.

Properties of Tor:

- Well-defined up to isomorphism. If $P_\bullet$ and $P'_\bullet$ are two different free resolutions of a module M, write $T_i(-)$ for $\operatorname{Tor}_i^R(M,-)$ computed via the resolution $P_\bullet$, and $T'_i(-)$ for $\operatorname{Tor}_i^R(M,-)$ computed via the resolution $P'_\bullet$. Then there are isomorphisms $\theta_N^i : T^i(N) \xrightarrow{\cong} T'^i(N)$ for all i, and $\theta_N^i \circ T^i(\alpha) = T'^i(\alpha) \circ \theta_N^i$ for all homomorphisms α. For all purposes, we can assume that functors $\operatorname{Tor}_i^R(M,-)$ are well-defined. Even better, we can use a resolution (exact sequence as above) where each F_i is merely *flat*, instead of free.
- The long exact sequence. If $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \to 0$ is a short exact sequence, then there is a long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(M,N') \xrightarrow{\operatorname{Tor}_2^R(M,\alpha)} \operatorname{Tor}_2^R(M,N) \xrightarrow{\operatorname{Tor}_2^R(M,\beta)} \operatorname{Tor}_2^R(M,N'') \to \operatorname{Tor}_1^R(M,N')$$
$$\xrightarrow{\operatorname{Tor}_1^R(M,\alpha)} \operatorname{Tor}_1^R(M,N) \xrightarrow{\operatorname{Tor}_1^R(M,\beta)} \operatorname{Tor}_1^R(M,N'') \to M \otimes_R N' \xrightarrow{M\otimes\alpha} M \otimes_R N \xrightarrow{M\otimes\beta} M \otimes_R N'' \to 0.$$

- Balancing of Tor (the magic trick). There are isomorphisms for each i, M, N:
$$\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^R(N,M).$$
That is, Tor can be computed by a free resolution of either argument!

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(8) Show that $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$, and $\operatorname{Tor}_0^R(M,\alpha)$ agrees with $M \otimes \alpha$.
(9) Compute $\operatorname{Tor}_1^{K[x]}(K[x]/(x), N)$ for an arbitrary $K[x]$-module N.
(10) Compute $\operatorname{Tor}_i^R(R/(x,y), R/(x))$ for $R = K[x,y]/(xy)$.
(11) Show that if R is Noetherian and M and N are finitely generated modules, then all of the modules $\operatorname{Tor}_i^R(M,N)$ are finitely generated.
(12) Let $(R,\mathfrak{m},k)$ be graded or local Noetherian, and M be finitely generated. Let $F_\bullet$ be a minimal free resolution of M. Show that the rank of F_i is the k-vector space dimension of $\operatorname{Tor}_i^R(M,k)$.
(13) Let $R = k[x_1,\dots,x_n]$ be a polynomial ring with the standard grading. Use the short exact sequences

$$0 \to R/(x_1,\dots,x_i) \xrightarrow{\cdot x_{i+1}} R/(x_1,\dots,x_i) \to R/(x_1,\dots,x_{i+1}) \to 0$$

to compute $\operatorname{Tor}_j^R(k,k)$. Conclude that k has a free resolution that ends at step at most j.
(14) Show that every finitely generated graded module over $R = k[x_1,\dots,x_n]$ has a free resolution that ends at step at most j.

After we learn about regular rings...

(15) Modify the last two problems to show that if $(R,\mathfrak{m})$ is a regular local ring, then every finitely generated R-module has a free resolution that ends at step at most j.
(16) Show that if M is a finitely generated module over $(R,\mathfrak{m})$, and x is a nonzerodivisor on M, then $\operatorname{pd}_R(M) = \operatorname{pd}_{R/xR}(M/xM)$.

Auslander-Buchsbaum-Serre Theorem: Let $(R,\mathfrak{m},k)$ be a local ring. The following are equivalent:

- R is regular.
- Every finitely generated R-module has a finite free resolution.
- k has a finite free resolution.

(17) Show the important corollary: if $(R,\mathfrak{m})$ is regular, and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_\mathfrak{p}$ is regular.