Matlis duality and $E_R(k)$ for rings with coefficient fields

Throughout, $(R, \mathfrak{m})$ will denote a local or graded noetherian ring with coefficient field k, so that the composition $k \hookrightarrow R \to R/\mathfrak{m}$ is an isomorphism.

Definition: For an R-module M, the *$\mathfrak{m}$-adically continuous k-linear homomorphisms from M to k* are

$$\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M, k) := \varinjlim \operatorname{Hom}_k(M/\mathfrak{m}^n M, k).$$

Before we get started, we recall some facts about Hom and limits:

- Given a directed system $(M_i)_{i\in\mathbb{N}} = \left(\cdots \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow \cdots \right)$,
$$\operatorname{Hom}_R\left(\varinjlim M_i, N\right) \cong \varprojlim \operatorname{Hom}_R(M_i, N).$$
- If M is finitely presented and given a directed system $(N_i)_{i\in\mathbb{N}}$, then
$$\operatorname{Hom}_R\left(M, \varinjlim N_i\right) \cong \varinjlim \operatorname{Hom}_R(M, N_i).$$
- In general, if $(M_i)_{i\in\mathbb{N}} = \left(\cdots \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \right)$ is an inverse system,
$$\operatorname{Hom}_R\left(\varprojlim M_i, N\right) \ncong \varinjlim \operatorname{Hom}_R(M_i, N)$$

---

0) Check that if M is an R-module, then $\left(\operatorname{Hom}_k(M/\mathfrak{m}^n M, k)\right)_{n\in\mathbb{N}}$ forms a directed system of R-modules, so $\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M, k)$ is an R-module.[1]

1) a) Show that $k \cong \operatorname{Hom}_R(R, k) \subseteq \operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(R, k)$ is an essential extension.

   b) Show that if M is a finitely generated R-module, then
$$\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M, k) \cong \operatorname{Hom}_R\left(M, \operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(R, k)\right).$$

   c) Show that $\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(R, k)$ is injective, and conclude that this is an injective hull of k.[2]

2) a) Let $S = k[x_1, \ldots, x_d]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, $T = S_\mathfrak{m}$. Show that $E_T(k)$ has a basis given by
$$(\underline{x}^{\underline{\alpha}})^* = \begin{cases} \underline{x}^{\underline{\beta}} \mapsto 1 & \underline{\beta} = \underline{\alpha} \\ \underline{x}^{\underline{\beta}} \mapsto 0 & \underline{\beta} \neq \underline{\alpha} \end{cases}$$
   with module structure given by
$$\underline{x}^{\underline{\gamma}} \cdot (\underline{x}^{\underline{\alpha}})^* = \begin{cases} (\underline{x}^{\underline{\alpha}-\underline{\gamma}})^* & \alpha_i \geqslant \gamma_i \text{ for all } i \\ 0 & \alpha_i < \gamma_i \text{ for some } i. \end{cases}$$

   b) Compare $E_T(k)$ to the module described in the first lecture.

[1] Yes, this is very quick.

[2] Hint: Show that for any ideal I and any n, there exists N such that $I/\mathfrak{m}^N \cap I \twoheadrightarrow I/\mathfrak{m}^n I$, $I/\mathfrak{m}^N \cap I \hookrightarrow R/\mathfrak{m}^N$.

c) Consider $\widehat{T} = k[\![x_1, \dots, x_n]\!]$, the completion of T above. Show that $E_{\widehat{T}}(k) \cong E_T(k)$.

d) If $R = \oplus_{i \in \mathbb{N}} R_i$ is graded, $R_0 = k$, and R is a finitely generated k-algebra, show that the graded dual

$$R^* = \bigoplus_i R_i^*, \qquad R_i^* = \operatorname{Hom}_k(R_i, k)$$

is an injective hull for k.

Definition: The *Matlis duality functor* is $(-)^\vee = \operatorname{Hom}_R(-, E_R(k))$. Note that this is an exact functor.

(3) a) If M is a finitely generated R-module, show that $M^\vee \cong \operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M, k)$.

b) If M is a module of finite length, show that $M^\vee \cong \operatorname{Hom}_k(M, k)$.

c) If $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M$, each M_i is a finite length module, and $M = \bigcup M_i$, we say that *M is a union of finite length submodules*. Show that if M is a union of finite length submodules, then $M^\vee \cong \operatorname{Hom}_k(M, k)$.

d) Show that if M is finitely generated, then $M^\vee$ is a union of finite length submodules.

e) If M is a finitely generated R-module, show that $M^{\vee\vee} \cong \widehat{M}$. In particular, $R^{\vee\vee} \cong \widehat{R}$.

(4) a) Come up with explicit examples of Matlis duals of nonregular rings and nonfree modules.

b) Say R is complete. Our recipe for $E_R(k)$ looks canonical. Explain why it is not.

c) Show that the Matlis duality functor is faithful: if $M \neq 0$, then $M^\vee \neq 0$.

d) Suppose R is a local ring containing a field, but that R does not have a coefficient field. Explain how to give an explicit description of $R^\vee$ anyway.

e) Can you give a uniform description for the duality functor for all R-modules in our setting?