# Worksheet on Mayer-Vietoris

(1) Let $R$ be a ring, $M$ an $R$-module, and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ ideals of $R$.

(a) Show that if $\mathfrak{a} \subseteq \mathfrak{c}$, there is an injective map $\Gamma_{\mathfrak{c}}(M) \to \Gamma_{\mathfrak{a}}(M)$, functorial in $M$.

(b) Show that

$$0 \to \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \to \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \to \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M)$$

is left exact.

(c) Show that if $R$ is noetherian, and $M = E$ is injective, then the sequence above is exact on the right.

(d) Show that if $R$ is noetherian, then there is a LES

$$0 \to \mathrm{H}^0_{\mathfrak{a}+\mathfrak{b}}(M) \to \mathrm{H}^0_{\mathfrak{a}}(M) \oplus \mathrm{H}^0_{\mathfrak{b}}(M) \to \mathrm{H}^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \mathrm{H}^1_{\mathfrak{a}+\mathfrak{b}}(M) \to \mathrm{H}^1_{\mathfrak{a}}(M) \oplus \mathrm{H}^1_{\mathfrak{b}}(M) \to \mathrm{H}^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \cdots$$

This is called the *Mayer-Vietoris sequence* of local cohomology.

(2) (a) Compute the cohomological dimension of $I = (x, y) \cap (u, v)$ in $K[x, y, u, v]$.

(b) Compute the cohomological dimension of $J = (x, y, z) \cap (u, v, w)$ in $K[x, y, z, u, v, w]$

(c) Based on the previous computations and the number of generators of the ideals, give a range of possible values for $\mathrm{ara}(I)$ and $\mathrm{ara}(J)$.

Definition: If $(R, \mathfrak{m}, k)$ is a local ring, the *punctured spectrum* of $R$ is the topological space $\mathrm{Spec}^\circ(R) := \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$.

(3) Let $(R, \mathfrak{m}, k)$ be a local ring.

(a) Prove that the (unpunctured) spectrum of $R$, $\mathrm{Spec}(R)$, is connected as a topological space.

(b) Prove that $\mathrm{Spec}^\circ(R)$ is disconnected as a topological space if and only if there exist ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ such that $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{(0)}$ and $\sqrt{\mathfrak{a} + \mathfrak{b}} = \mathfrak{m}$.

(c) Show that if $\mathrm{depth}(R) \geq 2$, then $\mathrm{Spec}^\circ(R)$ is connected.

(d) Show that if $R$ is Cohen-Macaulay, and $\mathrm{height}(I) > 1$, then $\mathrm{Spec}(R) \smallsetminus \mathcal{V}(I)$ is connected.

(4) We will also use another long exact sequence. Let $R$ be a noetherian ring, $I$ an ideal, and $x \in R$. Then for any $R$-module $M$,

$$0 \to \mathrm{H}^0_{I+(x)}(M) \to \mathrm{H}^0_I(M) \to \mathrm{H}^0_I(M_x) \to \mathrm{H}^1_{I+(x)}(M) \to \cdots.$$

(a) Show that $0 \to \Gamma_{I+(x)}(E) \to \Gamma_I(E) \to \Gamma_I(E_x) \to 0$ is exact for any injective module $E$.

(b) Prove the existence of the long exact sequence above.

(c) Show that if $I = (f_1, \ldots, f_t)$, then there is a short exact sequence of complexes

$$0 \to \check{C}^\bullet(\underline{f}; M_x)[1] \to \check{C}^\bullet(\underline{f}, x; M) \to \check{C}^\bullet(\underline{f}; M) \to 0.$$

(d) Give a second proof of the long exact sequence above.

(5) Compute $\operatorname{ara}(I)$ and $\operatorname{ara}(J)$ from #2.