Worksheet on Gorenstein rings

(1) Let $(R, \mathfrak{m}, k)$ be a local ring. Show[1] the following:

(a) R is Gorenstein if and only if $\widehat{R}$ is Gorenstein.

(b) If $x_1, \dots, x_i$ is a regular sequence, then R is Gorenstein if and only if $R/(x_1, \dots, x_i)$ is Gorenstein.

(c) If R is Gorenstein, and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is Gorenstein.

Definition: A local ring $(R, \mathfrak{m}, k)$ is a *complete intersection* if there is a surjection from a complete regular local ring S onto $\widehat{R}$ with the kernel generated by a regular sequence.

Fact: The following are equivalent:

- R is a complete intersection;
- For every[2] surjection from a complete RLR $S \twoheadrightarrow \widehat{R}$, the kernel is generated by a regular sequence;

and **IF** there exists a surjection from an RLR onto R itself,

- For every surjection from an RLR $S \twoheadrightarrow R$, the kernel is generated by a regular sequence.

(2) Show that if R is a complete intersection, then R is Gorenstein.

(3) Let K be a field. For each of the following rings R determine: Is R Gorenstein? Is R Cohen-Macaulay? Is R a complete intersection? Reuse your work from old worksheets and HW when convenient.

(a) $R = \dfrac{K[\![x,y]\!]}{(x^2, xy)}$.

(b) $R = \dfrac{K[\![x,y,z]\!]}{(xy, xz, yz)}$.

(c) $R = \dfrac{K[\![x,y,z]\!]}{(x^2, y^2, z^2, x(y-z), (x-y)z)}$.

(d) $R = \dfrac{K[X_{2\times 3}]_{\mathfrak{m}}}{I_2(X_{2\times 3})}$, where $\mathfrak{m}$ is the ideal generated by the entries of X.

(e) $R = K[x,y]^{(2)}_{\mathfrak{m}}$, where $\mathfrak{m}$ is the ideal generated by the positive degree forms.

(f) $R = K[x,y]^{(3)}_{\mathfrak{m}}$, where $\mathfrak{m}$ is the ideal generated by the positive degree forms.

(g) $R = K[x,y,z]^{(3)}_{\mathfrak{m}}$, where $\mathfrak{m}$ is the ideal generated by the positive degree forms.

[1] or note that we have already shown

[2] This quantifier is never vacuous by Cohen's Structure Theorem.

(4) Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be two complete Gorenstein local rings, with $R = S/I$.

(a) Use Local Duality to show that $\operatorname{Ext}^t_S(R, S) = \begin{cases} 0 & t < \dim(S) - \dim(R) \\ R & t = \dim(S) - \dim(R). \end{cases}$

(b) Suppose moreover that S is regular. Let $P_\bullet \to R$ be the minimal free resolution of R as an S-module. Show that, $P_\bullet \cong \operatorname{Hom}_S(P_\bullet, S)$.[3][4]

(c) With the same assumptions as in the previous part, show that $\operatorname{Tor}^S_i(R, M) \cong \operatorname{Ext}^i_S(R, M)$ for all S-modules M.

(5) A *numerical semigroup* is a subsemigroup S of $\mathbb{N}$; our convention is that $0 \in S$. A *numerical semigroup ring* is a ring of the form $K[S] := K[\{x^s \mid s \in S\}] \subseteq K[x]$. Assume that the GCD of the elements in S is 1. Then, there is a largest number $f_S \in \mathbb{N}$ such that $f_S \notin S$, called the *Frobenius number of* S.

(a) Show that $\mathrm{H}^1_{\mathfrak{m}}(K[S])$ is generated as a K vector space by $\{x^a \mid a \leq f_S \text{ and } a \notin S\}$.[5]

(b) Show that $K[S]$ is Gorenstein[6] if and only if

$$\{c \mid 0 \leq c \leq f_S, c \notin S\} = \{f_S - d \mid 0 \leq d \leq f_S, d \in S\}.$$

(c) Check this criterion with $K[\![x^3, x^7]\!]$, $K[\![x^3, x^5, x^7]\!]$, and $K[\![x^4, x^5, x^6]\!]$.

(6) Show that if R is a Gorenstein local ring, and M is a finitely generated R-module, then M has finite projective dimension if and only if M has finite injective dimension.

[3]To match up the indexing, $P_\bullet \cong [\operatorname{Hom}_S(P_\bullet, S)]^{d-\bullet}$

[4]You can use without proof the fact that any two minimal resolutions are isomorphic.

[5]Hint: Consider the ideal $J = x^{f_S+1}K[x]$ in $K[S]$.

[6]after localizing and/or completing at the ideal of positive degree elements