Computing local cohomology using the Čech complex

1) If $I = (f_1, \ldots, f_n)$ is an n-generated ideal, then

$$\mathrm{H}^n_I(M) = \check{H}^n(f_1, \ldots, f_n; M) = \text{ cohomology of } \left(\bigoplus_{i=1}^{n} M_{f_1 \cdots \widehat{f_i} \cdots f_n} \to M_{f_1 \cdots f_n} \to 0 \right),$$

so elements in the n-th local cohomology can be realized as equivalence classes of fractions.
Show that

$$\left[\frac{m}{f_1^t \cdots f_n^t} \right] \neq 0 \text{ in } H^n_I(M) \text{ if and only if } f_1^k \cdots f_n^k m \notin \left(f_1^{t+k}, \ldots, f_t^{t+k} \right) M \text{ for all } k \geq 0.$$

2) Let k be a field, $R = k[x_1, \ldots, x_n]$, and $\mathfrak{m} = (x_1, \ldots, x_n)$.

a) Show that $\left[\dfrac{\underline{x}^{\underline{\alpha}}}{x_1^t \cdots x_n^t} \right]$ is nonzero in $\mathrm{H}^n_{\mathfrak{m}}(R)$ if and only if $\underline{x}^{\underline{\alpha}} \notin (x_1^t, \ldots, x_n^t)$.

b) Compute $\mathrm{H}^n_{\mathfrak{m}}(R)$: give a k-basis and describe the R-module structure.

c) Use Čech cohomology to show that $\mathrm{H}^1_{(x,y)}(k[x,y]) = 0$. Beware that a potential element is represented by a pair of elements in R_x and R_y, and that problem #1 does not apply.

3) Let R and $\mathfrak{m}$ be as above, and $S = k[\underline{x}]^{(d)}$ be the subalgebra generated by the polynomials whose degrees are multiples of d. Let $\mathfrak{n}$ be its homogeneous maximal ideal (the ideal generated by all d-forms in S).

a) Show that $\mathrm{H}^i_{(x_1, \ldots, x_n)}(R) = \mathrm{H}^i_{(x_1^d, \ldots, x_n^d)}(R)$ for all i.

b) Show that $(R_{x_1^d \cdots x_i^d})^{(d)} = S_{x_1^d \cdots x_i^d}$ for all i.[1]

c) Show that $\check{C}^\bullet(x_1^d, \ldots, x_n^d; S) = \check{C}^\bullet(x_1^d, \ldots, x_n^d; R)^{(d)}$; i.e., this is the complex consisting of sums of elements whose degree is a multiple of d in $\check{C}^\bullet(x_1^d, \ldots, x_n^d; R)$.

d) Conclude that $\mathrm{H}^n_{\mathfrak{n}}(S) = \mathrm{H}^n_{\mathfrak{m}}(R)^{(d)}$. Find two linearly independent elements of highest degree in $\mathrm{H}^2_{(x^3, x^2y, xy^2, y^3)}(k[x^3, x^2y, xy^2, y^3])$.

[1] Given a graded module, we write $M^{(d)}$ for its d-th veronese submodule $\oplus_{k \in \mathbb{N}} M_{dk}$.

4) Let $T = \dfrac{k[x,y,u,v]}{(xu-yv)}$. Note that T admits an $\mathbb{N}^2$-grading via

$$\deg(x) = \begin{bmatrix}1\\0\end{bmatrix}, \deg(y) = \begin{bmatrix}0\\1\end{bmatrix}, \deg(u) = \begin{bmatrix}0\\1\end{bmatrix}, \deg(v) = \begin{bmatrix}1\\0\end{bmatrix};$$

since the defining equation is homogeneous with respect to this grading, we get a well-defined grading on T.

(a) Show that $\left(\dfrac{v}{x}, \dfrac{u}{y}\right)$ is a cocycle in the Čech complex $\check{C}^1(x,y;T)$.

(b) Show that the class $\left[\dfrac{v}{x}, \dfrac{u}{y}\right]$ of the cocycle in the previous part gives a nonzero class in $\mathrm{H}^1_{(x,y)}(T)$.

(c) Let $\eta_a = \left[\dfrac{v^{a-1}y^{a-1}}{x^a y^a}\right] \in \check{H}^2(x,y;T)$. Use the grading defined above to show that $\eta_a \neq 0$ in $\mathrm{H}^2_{(x,y)}(T)$.

(d) Show that each of the elements η_a is killed by the ideal $\mathfrak{m} = (x,y,u,v)$. Conclude that the socle of this local cohomology module (the submodule annihilated by the maximal ideal $\mathfrak{m}$) is infinite-dimensional.

(e) Congratulate yourself; you have disproven a conjecture of Grothendieck!

5) Let $R = k\begin{bmatrix}u & v & w\\ x & y & z\end{bmatrix}$ and $\mathfrak{p} = (uy - vx, uz - wx, vz - wy)$. Is $\mathrm{H}^3_{\mathfrak{p}}(R)$ nonzero?