Worksheet on Cohen-Macaulay rings

DEFINITION: A Noetherian local ring is **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$. A finitely generated module over a Noetherian local ring is a **Cohen-Macaulay module** if $\operatorname{depth}(M) = \dim(M)$. It is a **maximal Cohen-Macaulay module** if $\operatorname{depth}(M) = \dim(M) = \dim(R)$.

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(1) COHEN-MACAULAY RINGS ARE UNMIXED: Show that if M is a Cohen-Macaulay R-module, then $\operatorname{depth}(M) = \dim(R/\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Ass}(M)$. In particular, if R is a Cohen-Macaulay ring, then R has no embedded primes, and $\dim(R/\mathfrak{p})$ is the same for each $\mathfrak{p} \in \operatorname{Min}(R)$.

(2) Show that if $(R, \mathfrak{m})$ is Cohen-Macaulay, and $x_1, \dots, x_a$ is a regular sequence in R, then $R/(x_1, \dots, x_a)$ is Cohen-Macaulay.

(3) REGULAR SEQUENCES AND SYSTEMS OF PARAMETERS: Let $f_1, \dots, f_t$ be a sequence of elements in a ring $(R, \mathfrak{m})$. Show that, for the conditions
  (i) $f_1, \dots, f_t$ is a regular sequence;
  (ii) $\operatorname{ht}((f_1, \dots, f_i)) = i$ for all i;
  (iii) $\operatorname{ht}((f_1, \dots, f_t)) = t$;
  (iv) $f_1, \dots, f_t$ is part of a system of parameters for R,
we have (1)⇒(2)⇒(3)⇒(4) in general, and if R is Cohen-Macaulay, (4)⇒(1)[1], so all of the above are equivalent.

(4) COHEN-MACAULAYNESS BY SYSTEMS OF PARAMETERS: Let $(R, \mathfrak{m})$ be a Noetherian local ring. Show that the following are equivalent:
- R is Cohen-Macaulay;
- Some system of parameters of R is a regular sequence;
- Every system of parameters of R is a regular sequence.

(5) EXAMPLES:
  (a) Show that a regular local ring is Cohen-Macaulay. In particular, $K[\underline{x}]_{\underline{x}}$ is Cohen-Macaulay.
  (b) Show that $\dfrac{K[x,y,z]_{(x,y,z)}}{(x^2+y^3+z^7)}$ is Cohen-Macaulay. Even better, show that if R is a regular local ring, then $R/(f)$ is Cohen-Macaulay.
  (c) Show that every zero-dimensional local ring is Cohen-Macaulay.
  (d) Show that every one-dimensional local domain is Cohen-Macaulay.
  (e) Use a result above to show that $\dfrac{K[x,y]_{(x,y)}}{(x^2, xy)}$ is *not* Cohen-Macaulay.
  (f) Use a result above to show that $\dfrac{K[x,y]_{(x,y,z)}}{(xy, xz)}$ is *not* Cohen-Macaulay.

[1] Hint: Use #1 to show that f_1 is a nonzerodivisor, and use #2 for induction purposes.

(6) MORE EXAMPLES: Use #4 to determine which of the following are Cohen-Macaulay:
  (a) $\dfrac{K[x,y]_{(x,y)}}{(xy)}$.
  (b) $K[x^4, x^3y, xy^3, y^4]_{(x^4,x^3y,xy^3,y^4)}$.[2]
  (c) $\dfrac{K[X_{2\times 3}]_X}{I_2(X)}$.
  (d) $K\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \dfrac{\mathbb{Q}[u,v,x,y,z]}{(x^3+y^3+z^3)}$.

(7) (a) Let R be a Cohen-Macaulay local ring of dimension d, and $\mathfrak{p}$ be prime of height h. Show that there exists a regular sequence $f_1, \dots, f_h \in \mathfrak{p}$.
  (b) Let R be a Cohen-Macaulay local ring, and $\mathfrak{p}$ be prime. Show that $R_\mathfrak{p}$ is Cohen-Macaulay.

DEFINITION: A Noetherian local ring that may or may not be local is **Cohen-Macaulay** if $R_\mathfrak{m}$ is Cohen-Macaulay for all $\mathfrak{m} \in \operatorname{Max}(R)$. Equivalently, R is Cohen-Macaulay if $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

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(8) Show that any regular ring is Cohen-Macaulay. In particular, $\mathbb{Z}$, $K[\underline{x}]$, $\mathbb{Z}[\underline{x}]$ are Cohen-Macaulay.

(9) UNMIXEDNESS THEOREM: Let R be Cohen-Macaulay. Show that if $I = (f_1, \dots, f_r)$ has height r, then every associated prime of I has height r.[3] In particular, this holds true in a polynomial ring.

(10) DIMENSION FORMULA: Let R be a Cohen-Macaulay ring, and $\mathfrak{p} \subseteq \mathfrak{q}$ be primes. Then, $\operatorname{height}(\mathfrak{q}) - \operatorname{height}(\mathfrak{p}) = \dim(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})$. In particular, if $(R,\mathfrak{m})$ is Cohen-Macaulay and local, then $\dim(R) - \operatorname{height}(\mathfrak{p}) = \dim(R/\mathfrak{p})$.

(11) MIRACLE FLATNESS: Let $(R,\mathfrak{m}) \subseteq (S,\mathfrak{n})$ be a module-finite local inclusion of Noetherian local rings, with R regular. Then S is Cohen-Macaulay if and only if S is a free R-module.
  (a) Why does it suffice to show that, for a fixed regular system of parameters of R, $x_1, \dots, x_d$, that $x_1, \dots, x_d$ is a regular sequence on $S \iff S$ is a free R-module?
  (b) Show the "$\Leftarrow$" implication.
  (c) For the other direction, we induce on $d = \dim(R) = \dim(S)$. Do the case $d = 0$.
  (d) Fix a minimal generating set for S as an R-module, $s_1, \dots, s_t$. Show that if there is some nontrivial relation $r_1s_1 + \cdots + r_ts_t = 0$, then there is a nontrivial relation where not all of the r_i's lie in (x_d).
  (e) Consider what happens to a nontrivial relation nontrivial relation $r_1s_1 + \cdots + r_ts_t = 0$ when we pass to $R/(x_d)$, and obtain a contradiction.

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[2]Hint: $x^4 \cdot (xy^3)^2 = y^4 \cdot (x^3y)^2 \in (y^4)$.

[3]Hint: If not, take an embedded prime $\mathfrak{q}$, and contradict the fact that $(R/I)_\mathfrak{q}$ is CM.