Worksheet on regular sequences and depth

DEFINITION: A sequence of elements $x_1, \dots, x_t$ in a ring R is a **regular sequence** on a module M if

- x_1 is a nonzerodivisor on M,
- x_2 is a nonzerodivisor on M/x_1M,
- $\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$
- x_t is a nonzerodivisor on $M/(x_1, \dots, x_{t-1})M$, and
- $M \neq (x_1, \dots, x_t)M$.

A regular sequence $x_1, \dots, x_t$ is **maximal** if there is no regular sequence of the form $x_1, \dots, x_t, x_{t+1}$. When we don't specify a module M, we mean a regular sequence on R.

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(1) SOME FIRST REGULAR SEQUENCES.
 (a) We say that a sequence $x_1, \dots, x_t$ in a ring R is a **prime sequence** if each of the ideals (0), (x_1), (x_1, x_2), $\dots$, $(x_1, \dots, x_t)$ is prime and distinct. Show that a prime sequence is a regular sequence.
 (b) Show that the sequence of variables in a polynomial ring is a regular sequence.
 (c) Show that a sequence of monomials in a polynomial ring is a regular sequence if and only if no two of the monomials have a variable factor in common.

(2) Let $x, y \in R$ be a regular sequence on an R-module M. Show that, for any "coincidence" $xm = yn$ with $m, n \in R$, there is a "perfectly good explanation" $t \in M$ such that $n = xt$ and $m = yt$. Now, find a module over a ring with a coincidence that has no perfectly good explanation.

(3) PERMUTING REGULAR SEQUENCES.
 (a) In $K[x, y, z]$, show that $x, y(x-1), z(x-1)$ is a regular sequence, but $y(x-1), z(x-1), x$ isn't.
 (b) If $(R, \mathfrak{m})$ is Noetherian local, and $f, g \in \mathfrak{m}$, show that f, g is a regular sequence if and only if g, f is.
 (c) If $(R, \mathfrak{m})$ is Noetherian local, and $f_1, \dots, f_t \in \mathfrak{m}$, show that $f_1, \dots, f_t$ is a regular sequence if and only if any permutation of it is.
 (d) Explain what small change you need to make to your argument to show the same statement for homogeneous elements in a graded ring.

(4) MAXIMAL REGULAR SEQUENCES: If $(R, \mathfrak{m})$ is Noetherian local, and M is finitely generated, any two maximal regular sequences on M have the same length.
 (a) Why do maximal regular sequences even exist in this case?
 (b) Take $a_1, \dots, a_n$ to be a maximal regular sequence on M with n minimal, and $b_1, \dots, b_n$ another regular sequence on M. Explain why it suffices to show that every element of $\mathfrak{m}$ is a zerodivisor on $M/(b_1, \dots, b_n)$. We will induce on n.
 (c) For the case $n = 1$, find an element $m \in M$ such that $\mathfrak{m}m \subseteq a_1M$, so $b_1m = a_1m'$ for some m'. Show that $\mathfrak{m}$ annihilates $\overline{m'} \in M/(b_1)M$,[1] and conclude this case.
 (d) Now suppose the claim is true for all modules in which there is a maximal regular sequence of length $< n$. Take an element $c \in \mathfrak{m}$ that is a nonzerodivisor on $M/(a_1, \dots, a_i)M$ and $M/(b_1, \dots, b_i)M$ for all $i < n$. Why does such an element exist?
 (e) Show that $a_1, \dots, a_{n-1}, c$ is a maximal regular sequence on M, that $c, a_1, \dots, a_{n-1}$ is a maximal regular sequence on M, apply the induction hypothesis to $M/(c)M$, and conclude the proof.

[1] Hint: Go deeper into the rabbithole—multiply by a_1.

DEFINITION: Let $(R, \mathfrak{m})$ be a local ring. The **depth** of a module M is the maximal length of a regular sequence on M, denoted $\mathrm{depth}(M)$. The **depth** of R is $\mathrm{depth}(R)$.

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(5) Let $(R, \mathfrak{m})$ be a Noetherian local ring. Show that $\mathrm{depth}(R) = 0 \Longleftrightarrow \mathfrak{m} \in \mathrm{Ass}(R)$.

(6) What is the depth of $\mathbb{Z}_{(p)}$? What is the depth of $K[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$, for K a field?

(7) Show that if $x_1, \dots, x_a$ is a regular sequence on M, then $\mathrm{depth}(M) = a + \mathrm{depth}(M/(x_1, \dots, x_a)M)$, and that this is the same whether we consider this as an R-module or as an $R/(\underline{x})$-module.

(8) Show that if $x_1, \dots, x_a$ is a regular sequence in a Noetherian ring R, then $\mathrm{height}((x_1, \dots, x_a)) = a$. Conclude that $\mathrm{depth}(S) \leq \dim(S)$ for S Noetherian local.

(9) What is the depth of
  (a) $K[x, y]_{(x,y)}/(x^2, xy)$?
  (b) $K[x, y, z]_{(x,y,z)}/(xy, xz)$?
  (c) $K[x, y, z]_{(x,y,z)}/(x^2 - yz)$?
  (d) $\dfrac{K[u, v, x, y]_{(u,v,x,y)}}{(xu, xv, yu, yv)}$

(10) DEPTH AND DIMENSION OF ASSOCIATED PRIMES: If $(R, \mathfrak{m})$ is Noetherian local, and $M \neq 0$ is finitely generated, then $\mathrm{depth}(M) \leq \min\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(M)\}$.
  (a) For the proof, fix $\mathfrak{p} \in \mathrm{Ass}(M)$. We may as well assume that $\mathrm{depth}(M) > 0$, so there is some nonzerodivisor $x \in \mathfrak{m}$ on M.
  (b) Take z such that $Rz \subseteq M$ is a maximal element in the set $\{Ra \subseteq M \mid \mathfrak{p}(Ra) = 0\}$. Show that $\overline{z} \in M/(x)M$ is nonzero, and that $\mathfrak{p} + (x) \subseteq \mathrm{ann}(\overline{z})$.
  (c) Show that there is some $\mathfrak{q} \in \mathrm{Ass}(M/(x)M)$ such that $\mathfrak{q} \supsetneq \mathfrak{p}$. Conclude the proof.