Worksheet supplement on minimal primes

DEFINITION:

- A **monomial ideal** in a polynomial ring $R = K[x_1, \ldots, x_n]$ is an ideal generated by monomials.
- A monomial $\prod_i x_i^{a_i}$ is **squarefree** if $a_i \in \{0, 1\}$ for all i.
- A **squarefree monomial ideal** is an ideal generated by squarefree monomials.
- A **simplicial complex** Δ on a set S is a collection of subsets of S (i.e., $\Delta \subseteq \mathcal{P}(S)$) such that $G \in \Delta$ and $F \subseteq G$ implies $F \in \Delta$.

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(1) Show that if

$$I = \left(\prod_i x_i^{a_{1,i}}, \ldots, \prod_i x_i^{a_{t,i}} \right)$$

is a monomial ideal in $K[x_1, \ldots, x_n]$, then

$$\sqrt{I} = \left(\prod_i x_i^{\overline{a_{1,i}}}, \ldots, \prod_i x_i^{\overline{a_{t,i}}} \right),$$

where $\overline{a_{j,i}} = 1$ if $a_{j,i} > 0$ and 0 if $a_{j,i} = 0$.

(2) Show that a monomial ideal is radical if and only if it is a squarefree monomial ideal.

(3) Show that the rule

$$I \quad \mapsto \quad \{S \subseteq [n] \mid \prod_{i \in S} x_i \notin I\}$$

induces a containment-reversing bijection[1] between squarefree monomial ideals in $K[x_1, \ldots, x_n]$ and simplicial complexes on the set $[n] := \{1, \ldots, n\}$.

(4) Given a squarefree monomial ideal I, describe the minimal primes of I in terms of the combinatorics of the simplicial complex described in the previous problem.

(5) Express the following ideals as intersections of prime ideals:
  (a) $I = (xy, xz, yz)$
  (b) $I = (xu, xv, yu, yv)$,
  (c) $I = (xyz, ux, uy, vy, vz)$,
  (d) $I = (uyz, vxz, uvz, wxy, uwy, vwx)$.

[1] DEFINITION: The **Stanley-Reisner ideal** of a simplicial complex is the ideal obtained by the inverse map to the one stated above.