# Worksheet on minimal and associated primes

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DEFINITION: If $I$ is an ideal, the **minimal primes** of $I$ are the primes $\mathfrak{p} \in V(I)$ that are minimal with respect to containment. That is,

$$\mathrm{Min}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I \text{ and } I \subseteq \mathfrak{q} \subseteq \mathfrak{p} \Rightarrow \mathfrak{q} = \mathfrak{p} \text{ for } \mathfrak{q} \in \mathrm{Spec}(R)\}.$$

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(0) Recall/reconvince yourself of the following facts:
  (a) $\mathfrak{p} \in V(I) \Rightarrow \exists \mathfrak{q} \in \mathrm{Min}(I) : \mathfrak{p} \supseteq \mathfrak{q}$.
  (b) $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p}$.

(1) Let $R$ be *Noetherian*, and $I$ be an ideal. Show that $\mathrm{Min}(I)$ is a finite set.[1]

(2) Use the previous problem to show that, in a *Noetherian* ring $R$, the following hold:
  (a) For any radical ideal $I$, there is a finite set of primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ such that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$.
  (b) A subset of $\mathrm{Spec}(R)$ is closed if and only if it is a finite union of upper poset intervals. Thus, the poset structure determines the topology.
  (c) For any affine variety $Y = Z_K(I) \subseteq K^n$, there is a finite set of primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ such that $Y = Z_K(\mathfrak{p}_1) \cup \cdots \cup Z_K(\mathfrak{p}_t)$.

(3*) Find a ring $R$ and an ideal $I$ with infinitely many minimal primes.

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DEFINITION: If $M$ is an $R$-module, the **support** of $M$ is $\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

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(4) (a) Show that the support of any module is specialization closed.
  (b) Show that the support of the cyclic module $R/I$ is $V(I)$.
  (c) Show that if $M$ is *finitely generated*, then the support of $M$ is $V(\mathrm{ann}_R(M))$.
  (d) Show that if we don't assume that $M$ is finitely generated, then $\mathrm{Supp}(M)$ may not equal $V(\mathrm{ann}_R(M))$.

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DEFINITION: Let $R$ be a ring, and $M$ a module. We say that $\mathfrak{p} \in \mathrm{Spec}(R)$ is an **associated prime** of $M$ if $\mathfrak{p} = \mathrm{ann}_R(m)$ for some $m \in M$. Equivalently, $\mathfrak{p}$ is associated to $M$ if there is an injective homomorphism $R/\mathfrak{p} \hookrightarrow M$. We write $\mathrm{Ass}_R(M)$ for the set of associated primes of $M$.

If $I$ is an ideal, by the **associated primes** of $I$ we (almost always) mean the associated primes of $R/I$; but we'll try to write $\mathrm{Ass}_R(R/I)$.

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(5) Show that if $\mathfrak{p}$ is prime, then $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

(6) Show that if $R = K[x, y]$, and $I = (x^2, xy)$, then $\{(x), (x, y)\} \subseteq \mathrm{Ass}_R(R/I)$.

(7) Let $R$ be a *Noetherian* ring, and $M$ be a module. Show that the set of ideals $S = \{I \mid \exists m \in M : I = \mathrm{ann}_R(m)\}$ has a maximal element, and any such maximal element is prime.

[1]Hint: If there is an $I$ such that $\mathrm{Min}(I)$ is infinite, then among the ideals with this property there is a maximal one (why?). It's not prime...

(8) Let $R$ be a *Noetherian* ring, and $M$ an $R$-module. Show that $M \neq 0 \Longleftrightarrow \operatorname{Ass}_R(M) \neq \varnothing$.

(9) Let $R$ be a *Noetherian* ring, and $M$ an $R$-module. An element of $R$ is a *zerodivisor* on $M$ if $rm = 0$ for some $m \in M \smallsetminus 0$. Show that

$$\{\text{zerodivisors on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p} .$$

(10) Show that $0 \to L \to M \to N \to 0$ is exact, then $\operatorname{Ass}(L) \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(L) \cup \operatorname{Ass}(N)$.

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DEFINITION: A **finite filtration** of a module $M$ is a finite ascending chain of submodules:

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M.$$

A finite filtration as above is a **prime filtration** if $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$ for each $i$.

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(11) Let $R$ be a *Noetherian* ring, and $M$ a *finitely generated* module. Show that $M$ has a prime filtration.

(12) Show that if $R$ is *Noetherian*, and $M$ a *finitely generated* module, then $|\operatorname{Ass}_R(M)| < \infty$.

(13*) Show that if we don't assume that $M$ is finitely generated, then *any* subset of $\operatorname{Spec}(R)$ is $\operatorname{Ass}_R(M)$ for some module.

(14*) Find a ring $R$ and a module $M \neq 0$ such that $\operatorname{Ass}_R(M) = \varnothing$.

(15) Show that if $R$ is *Noetherian*, $M$ is an $R$-module, and $W$ is a multiplicative set, then

$$\operatorname{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R(M) \text{ and } \mathfrak{p} \cap W = \varnothing\}.$$

(16) Show that if $R$ is *Noetherian*, then $\operatorname{Min}(I) \subseteq \operatorname{Ass}_R(R/I)$.