Worksheet on primary decomposition

Definition: An ideal $\mathfrak{q}$ in a ring R is **primary** if $xy \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n \in \mathbb{N}$.
An ideal $\mathfrak{q}$ is **$\mathfrak{p}$-primary** if $\mathfrak{q}$ is primary and $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

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(1) (a) Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. So, a primary ideal is $\mathfrak{p}$-primary for some prime $\mathfrak{p}$.
(b) If R is a PID, show that $\mathfrak{q} = (a)$ is primary if and only if it is generated by a power of an irreducible element.
(c) Show that in $R = \mathbb{C}[x, y, z]$, the ideal $\mathfrak{q} = (x^2, xy)$ has $\sqrt{\mathfrak{q}} = (x)$ prime, but $\mathfrak{q}$ is *not* primary. Thus, the converse of (a) fails.
(d) Show that if $\mathfrak{q}$ is an ideal such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$ is a *maximal* ideal, then $\mathfrak{q}$ is $\mathfrak{m}$-primary.[1] Thus, a limited converse of (a) holds.

(2) Equivalent characterizations. Show that the following are equivalent:
- $\mathfrak{q}$ is primary.
- Every zerodivisor in $R/\mathfrak{q}$ is nilpotent.[2]
- $\mathrm{Ass}(R/\mathfrak{q})$ is a singleton.
- $\mathfrak{q}$ has one minimal prime, and no embedded primes.

(3) Equivalent characterizations. Show that the following are equivalent:
- $\mathfrak{q}$ is $\mathfrak{p}$-primary.
- $\mathrm{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$.
- $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}$ is saturated with respect to the multiplicative set $(R \smallsetminus \mathfrak{p})$.
- $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}R_{\mathfrak{p}} \cap R = \mathfrak{q}$.

(4) Show that if $I_1, \dots, I_t$ are ideals, then $\mathrm{Ass}(R/(\bigcap_{j=1}^t I_j)) \subseteq \bigcup_{j=i}^t \mathrm{Ass}(R/I_j)$. In particular, a finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.

Definition: A **primary decomposition** of an ideal I is an expression of the form $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, with each $\mathfrak{q}_i$ primary. That is, a primary decomposition is an expression of an ideal as a finite intersection of primary ideals. A **minimal primary decomposition** of an ideal I is a primary decomposition as above in which $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for $i \neq j$, and $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$ for all i.

By the previous problem, we can turn a primary decomposition into a minimal one by combining the terms with the same radical, then removing redundant terms.

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(5) Show that if R is a PID, any ideal of R can be written as an intersection of primary ideals in an essentially unique way. What does this look like in $\mathbb{Z}$?

(6) What is the minimal primary decomposition of a radical ideal in a Noetherian ring?

Existence Theorem[3]: If R is Noetherian, then every ideal admits a (minimal) primary decomposition.

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[1] You might find it easier to do #2 first and return to this.

[2] Hint: What is the set of all elements of R that are nilpotents mod $\mathfrak{q}$? What is the set of all elements of R that are zerodivisors mod $\mathfrak{q}$?

[3] The is the Noether-Lasker Theorem. For polynomial rings, this was first shown by 1894–1921 world chess champion Emanuel Lasker. Emmy Noether showed this for the class of rings named after her. Both were Ph.D. students of Hilbert. The proof has steps that can be slow, so we've moved it to the end.

First Uniqueness Theorem: If I is an ideal in a Noetherian ring R, then for any minimal primary decomposition of I, $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, we have $\{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_t}\} = \operatorname{Ass}(R/I)$. In particular, this set is the same for all minimal primary decompositions of I.[4]

Second Uniqueness Theorem: If I is an ideal in a Noetherian ring R, then for any minimal primary decomposition of I, $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, the set of **minimal components** $\{\mathfrak{q}_i \mid \sqrt{\mathfrak{q}_i} \in \operatorname{Min}(R/I)\}$ is the same. In particular, for $\sqrt{\mathfrak{q}_i} \in \operatorname{Min}(R/I)$, we have $\mathfrak{q}_i = IR_{\sqrt{\mathfrak{q}_i}} \cap R$.

(7) Proof of First Uniqueness Theorem:
  (a) Show the containment $\operatorname{Ass}(R/I) \subseteq \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_t}\}$ in the first uniqueness theorem.
  (b) Let $I_j = \bigcap_{i \neq j} \mathfrak{q}_i$. Show that there is a an element $x_j \in I_j \smallsetminus I$ such that $\overline{x_j} \in I_j/I$ has a prime annihilator, and that this annihilator must contain $\mathfrak{p}_j := \sqrt{\mathfrak{q}_j}$.
  (c) Use the definition of primary to show that the annihilator of any element of I_j/I is contained in $\mathfrak{p}_j$.
  (d) Conclude the proof of the First Uniqueness Theorem.

(8) Prove the Second Uniqueness Theorem.

(9) Find two different primary decompositions for (x^2, xy) in $\mathbb{C}[x, y]$.

(10*) If $\mathfrak{q}$ is $\mathfrak{p}$-primary, set $\ell\ell_{\mathfrak{p}}(\mathfrak{q})$ to be $\min\{n \mid \mathfrak{p}^n \subseteq \mathfrak{q}\}$. Show that, for an ideal I in a Noetherian ring R, there is a unique minimal primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ such that for any other minimal primary decomposition $I = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_t$, where WLOG $\sqrt{\mathfrak{q}_i} = \sqrt{\mathfrak{q}'_i} =: \mathfrak{p}_i$ for all i, we have
  - $\ell\ell_{\mathfrak{p}_i}(\mathfrak{q}_i) \leq \ell\ell_{\mathfrak{p}_i}(\mathfrak{q}'_i)$, and
  - $\ell\ell_{\mathfrak{p}_i}(\mathfrak{q}_i) = \ell\ell_{\mathfrak{p}_i}(\mathfrak{q}'_i) \Rightarrow \mathfrak{q}_i \subseteq \mathfrak{q}'_i$.

Definition: If $\mathfrak{p}$ is a prime ideal in a ring R, the nth **symbolic power** of $\mathfrak{p}$ is $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

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(11) Let R be Noetherian, and $\mathfrak{p}$ a prime ideal of R. Show that the following are all equal to $\mathfrak{p}^{(n)}$:
  - The unique smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$; in particular, $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$.
  - The (well-defined) $\mathfrak{p}$-primary component in a primary decomposition of $\mathfrak{p}^n$.
  - $\{f \in R \mid rf \in \mathfrak{p}^n \text{ for some } r \notin \mathfrak{p}\}$.

(12) Show that $(x, y)^{(2)} = (x, y)^2$ in $K[x, y, z]$, but $(x, y)^{(2)} \neq (x, y)^2$ in $K[x, y, z]/(y^2 - xz)$.

(13*) Let $X = X_{3\times 3}$ be a 3×3 matrix of indeterminates, and $K[X]$ be a polynomial ring over a field K. Let $\mathfrak{p} = I_2(X)$ be the ideal generated by 2×2 minors of X. Show that $\det(X) \in \mathfrak{p}^{(2)} \smallsetminus \mathfrak{p}^2$.

(14*) Proof of Existence Theorem:
  (a) An ideal $\mathfrak{q}$ is **indecomposable** if $\mathfrak{q} = I \cap J \Rightarrow \mathfrak{q} = I$ or $\mathfrak{q} = J$. Show that if R is Noetherian, then every ideal of R can be expressed as a finite intersection of irreducible ideals.
  (b) Show that if R is Noetherian, then any irreducible ideal is primary.[5] Conclude the proof of the existence theorem for primary decompositions.

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[4]An analogue of this holds for ideals in nonnoetherian rings that have a primary decomposition; however, the primes occurring are not necessarily the associated primes, but a variant on this definition.

[5]If $\mathfrak{q}$ is not primary, take $xy \in \mathfrak{q}$, $x \notin \mathfrak{q}$, $y \notin \sqrt{\mathfrak{q}}$, and consider the ideals $\mathfrak{q} : y^n$.