Worksheet on the spectrum of a ring

DEFINITION: The **spectrum**, or **prime spectrum** of a ring R is the set of prime ideals of R, denoted $\mathrm{Spec}(R)$.

The spectrum of a ring is enriched with two other structures:

- $\mathrm{Spec}(R)$ is a *poset* under containment: $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \subseteq \mathfrak{q}$.
- $\mathrm{Spec}(R)$ is a *topological space* with closed sets $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I\}$.

The **maximal spectrum** of a ring R is the set of maximal ideals of R, denoted $\mathrm{Max}(R)$. We give it the subspace topology induced from the inclusion $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$. That is, the closed subsets of $\mathrm{Max}(R)$ are the subsets of the form $V_{\mathrm{Max}}(I) := \{\mathfrak{m} \in \mathrm{Max}(R) \mid \mathfrak{m} \supseteq I\}$. Note that the poset structure on $\mathrm{Max}(R)$ is trivial.

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(1) SOME FIRST EXAMPLES OF PRIME SPECTRA:
  a) What are the elements of $\mathrm{Spec}(\mathbb{Z})$? Draw a (partial) picture of $\mathrm{Spec}(\mathbb{Z})$ as a poset. What are the closed subsets?
  b) What are the elements of $\mathrm{Spec}(\mathbb{C}[x])$? Draw a (partial) picture of $\mathrm{Spec}(\mathbb{C}[x])$ as a poset. What are the closed subsets?

(2) Let K be an algebraically closed field, and $R = K[\underline{x}]$. What does the Nullstellensatz say about the points of $\mathrm{Max}(R)$? What are the closed subsets of $\mathrm{Max}(R)$?

(3) Let R be a ring, and I an ideal. Find a natural homeomorphism between $V(I) \subseteq \mathrm{Spec}(R)$ (with the subspace topology) and $\mathrm{Spec}(R/I)$. Now, draw a picture of $\mathrm{Spec}(\mathbb{C}[x,y]/(xy))$ as a poset.

(4*) Let X be a compact manifold, and $R = \mathcal{C}(X, \mathbb{R})$ be the ring of continuous functions from X to $\mathbb{R}$.
  a) Show that if $I \subsetneq R$ is an ideal, then there is some $x \in X$ such that $f(x) = 0$ for all $f \in I$.
  b) Find a bijection between $\mathrm{Max}(R)$ and X.
  c) Show that your bijection takes closed subsets of $\mathrm{Max}(R)$ to closed subsets of X.
  d) Show that your bijection is a homeomorphism.

(5) Let $R = \mathbb{C}\{x, y\}$ be the ring of complex power series in $\mathbb{C}[\![x, y]\!]$ that converge on some ball containing the origin.
  a*) Show that if $f(0,0) \neq 0$, then f is a unit in R. Conclude that R has a unique maximal ideal (x, y).
  b*) Show that every prime $\mathfrak{p} \in \mathrm{Spec}(R)$ other than (0) and (x, y) is principal.
  c) Draw a rough picture of $\mathrm{Spec}(R)$ as a poset.
  d) Can you give a geometric interpretation for $\mathrm{Spec}(R)$? Why is this a much more interesting object that $\mathrm{Max}(R)$?

(6) Let R be a ring.
  a) Show that if I, J are ideals, then $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.[1]
  b) Show that if $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals, then $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V\big(\sum_{\lambda \in \Lambda} I_\lambda\big)$.
  c) Conclude that what we claimed was a topology is legitimately a topology!
  d) Show that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, the *upper poset interval determined by* $\mathfrak{p}$.

[1] I recommend $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ) \subseteq V(I) \cup V(J)$.

(7) Poset structure of Spec vs topological structure

a) How can you recover the poset structure of $\mathrm{Spec}(R)$ from the topological structure of $\mathrm{Spec}(R)$?

b) Show that every closed subset $X \subseteq \mathrm{Spec}(R)$ is *specialization-closed*; i.e., $\mathfrak{p} \in X$ and $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mathfrak{q} \in X$.

c) Find a ring R and a specialization-closed subset $X \subseteq \mathrm{Spec}(R)$ that is not closed.

d) Show that every element of $V(I)$ contains a minimal element of $V(I)$. Conclude that every closed set is the union of upper intervals determined by its minimal elements.

e) Conversely, show that every union of *finitely many* upper intervals is closed.[2]

(8*) The "Vsatz": Let R be a ring, and I, J be ideals. One has $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$. Hence, there is an order-preserving bijection between radical ideals of R and closed subsets of $\mathrm{Spec}(R)$.

Definition: If $\varphi : R \to S$ is a ring homomorphism, the **induced map on spectra** is the map $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$. We sometimes write $\mathfrak{q} \cap R$ for $\varphi^{-1}(R)$; this is honest when φ is an inclusion map.

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(9) Show that φ^* is well-defined and continuous.

(10) Find a pair of rings R, S, and a ring homomorphism $\varphi : R \to S$ for which φ^* does not restrict to a function from $\mathrm{Max}(S) \to \mathrm{Max}(R)$.

(11) In this problem, we will show that if R and S are finitely generated algebras over a field K, then φ^* restricts to a map from $\mathrm{Max}(S) \to \mathrm{Max}(R)$.

a) Show that if $A \to B$ is integral, then every element of B has a nonzero multiple in A.

b) Show that if K is a field, and A is a domain that is integral over K, then A is also a field.

c) Prove the statement.

(12) Let K be an algebraically closed field, and $R = K[x_1, \dots, x_m]/I$ and $S = K[y_1, \dots, y_n]/J$ be finitely generated K-algebras.

a) Show that for every K-algebra map $f : R \to S$, there is a map $F : K[x_1, \dots, x_m] \to K[y_1, \dots, y_n]$ given by $(x_1, \dots, x_m) \mapsto (f_1(\underline{y}), \dots, f_m(\underline{y}))$ for some polynomials $f_1, \dots, f_m$ such that the following diagram commutes:

$$\begin{array}{ccc} K[x_1, \dots, x_m] & \xrightarrow{F} & K[y_1, \dots, y_n] \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & S. \end{array}$$

b) In the context of the previous part, show that the map $\Phi : K^n \to K^m$ given by $\Phi(a_1, \dots, a_n) = (f_1(\underline{a}), \dots, f_m(\underline{a})))$ restricts to a map $\Phi : Z_K(J) \to Z_K(I)$.

c) Again, with the same notation, show that $f^*(\mathfrak{m}_{\underline{a}}) = \mathfrak{m}_{\Phi(\underline{a})}$.

d) If $\alpha : \mathbb{C}[x, y, z] \to C[x, y, z]$ is given by $\alpha(x) = x^2 - y, \alpha(y) = xz, \alpha(z) = 1 - z$, what is $\alpha^*((2x - 1, 3x + iy - \pi z, z))$?

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[2] We close the gap between these counterpart statements for a certain class of rings soon.