Worksheet on finiteness of normalizations

DEFINITION: If $K \subseteq L$ is a module-finite extension of fields, the **trace map** from L to K is

$$\mathrm{Tr}_{L/K}(l) = \text{trace of } K\text{-linear map } L \xrightarrow{\cdot l} L.$$

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(1) PROPERTIES OF TRACE: Prove the following properties.
  (a) The map $\mathrm{Tr}_{L/K} : L \to K$ is K-linear.
  (b) $\mathrm{Tr}_{L/K}(x) = [L : K]x$ for $x \in K$.
  (c) $-\mathrm{Tr}_{K(x)/K}(x)$ is the second coefficient of the minimal polynomial of x over K.
  (d) If $K \subseteq L \subseteq M$ are fields, then $\mathrm{Tr}_{M/K} = \mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{M/L}$.

(2) Show that if $K \subseteq L$ is a module-finite extension of fields, R is a normal subring of K, and $x \in L$ is integral over R, then $\mathrm{Tr}_{L/K}(x) \in R$.

FINITENESS THEOREM 1: Let R be a Noetherian normal domain, K its fraction field, and L a module-finite separable extension field of K. The integral closure S of R in L is module-finite over R, and hence Noetherian.

(3) PROOF OF FINITENESS THEOREM 1 IN CHARACTERISTIC ZERO: Suppose K has characteristic zero.
  (a) Show that there is a K vector space basis $\{s_1, \dots, s_d\}$ for L consisting of elements in S.
  (b) Show that there is a K vector space basis $\{t_1, \dots, t_d\}$ for L such that $\mathrm{Tr}_{L/K}(s_i t_j) = 1$ if $i = j$ and 0 if $i \neq j$.
  (c) Show that $S \subseteq Rt_1 + \cdots + Rt_d$.
  (d) Conclude the proof of the theorem in characteristic zero.

(4) Show that if K is a field of characteristic zero or $p > 0$, and L is a module-finite *separable* extension field of K, with L then there is some $l \in L$ such that $\mathrm{Tr}_{L/K}(l) \neq 0$.

(5) Prove Finiteness Theorem 1 above in general.

FINITENESS THEOREM 2: Let R be a domain that is algebra-finite over a field k. Let K be the fraction field of R, and L be a module-finite separable extension field of K. Then the integral closure of R in L is module-finite over R.

In particular, if R is a domain that is algebra-finite over a field k, the normalization of R is module-finite over R.[1]

(6) Prove Finiteness Theorem 2 in the case k has characteristic zero.[2]

(7) Prove Finiteness Theorem 2 in general.

[1]This is a theorem of Emmy Noether.

[2]Hint: Replace R by a Noether normalization, and apply Finiteness Theorem 1.

Counterexample to Finiteness Theorem 1 in lieu of separability: Let $t_0, t_1, t_2, t_3, \ldots$ be an infinite sequence of indeterminates over $\mathbb{F}_p$. Let $K_0 = \mathbb{F}_p(t_0^p, t_1^p, t_2^p, t_3^p, \ldots)$, and

$$K_0 \subseteq K_1 := K_0(t_0) \subseteq K_2 := K_1(t_1) \subseteq \cdots \subseteq K := \bigcup_{n \in \mathbb{N}} K_n.$$

Let $R = \bigcup_{n \in \mathbb{N}} K_n[\![x]\!] \subseteq K[\![x]\!]$. Let $u = \sum_{i=0}^{\infty} t_i x^i$. Then the integral closure of R in the fraction field of R adjoined u is not module-finite over R.[3]

(8) Proof of counterexample:
- (a) Show that $R \neq K[\![x]\!]$, but $(K[\![x]\!])^p \subseteq R$.
- (b) Show that $\operatorname{frac}(R)(u)$ is a module-finite extension field of $\operatorname{frac}(R)$.
- (c) Show that every element of R is a equal to a multiple of x times a unit in R.
- (d) Show that R is a Noetherian normal domain.
- (e) For $i \in \mathbb{N}$, set $u_i = \sum_{j=0}^{\infty} t_{i+j} x^j$. Show that $u_i \in \operatorname{frac}(R)(u)$, and that u_i is integral over R.
- (f) Show that, for all $i \in \mathbb{N}$, $u_i \notin Ru_0 + Ru_1 + \cdots + Ru_{i-1}$.
- (g) Conclude the proof of the counterexample.

(9) Show that, in the previous counterexample, the ring $S = R[u]$ is a Noetherian domain such that the normalization of S is not module-finite over S.

[3]This is an example of M. Nagata.