Homework #4

Please write up and turn in at least *four* of the following problems at the beginning of class Monday, April 8.

(1) Find primary decompositions and compute the cohomological dimensions of
$$I = (u^2 - x^2, v^2 - y^2, w^2 - z^2) \subseteq K[u, v, w, x, y, z]$$
and
$$J = (u^2 - x^2, uv - xy, uw - xz, v^2 - y^2, vw - yz, w^2 - z^2) \subseteq K[u, v, w, x, y, z],$$
where K is a field of characteristic not equal to two.

(2) Let R be a noetherian ring and I an ideal of R. Let $c = \mathrm{cd}(I, R)$. Show that for any R-module M, $\mathrm{H}^c_I(M) \cong \mathrm{H}^c_I(R) \otimes_R M$.

(3) Show that $\mathrm{H}^3_{I_2(X)}(\mathbb{Z}[X_{2\times 3}])$ is a $\mathbb{Q}$-vector space.

(4) Let R be Cohen-Macaulay. We showed in an earlier worksheet that for $\mathfrak{p} \subset R$ is prime, then $\mathrm{depth}_{\mathfrak{p}}(R) = \mathrm{height}(\mathfrak{p})$. Use this and the Mayer-Vietoris sequence to show that for general ideals $I \subset R$, $\mathrm{depth}_I(R) = \mathrm{height}(I)$.

(5) Show that if R is Cohen-Macaulay of dimension at least two, and I has height at least 1, then $\mathrm{Spec}(R) \smallsetminus \mathcal{V}(I)$ is connected.

(6) Compute the arithmetic rank of $I = (xu, xv, yu, yvz) \subseteq K[u, v, x, y, z]$, where K is a field.

(7) For two local rings $(R, \mathfrak{m}, k)$, $(S, \mathfrak{n}, k)$ with the same residue field k (i.e., with a fixed isomorphism between the residue fields), the *fiber product* of R and S over k is
$$R \times_k S := \{(r, s) \in R \times S \mid r + \mathfrak{m} = s + \mathfrak{n} \text{ in } k\}.$$
Show that such a ring $R \times_k S$ is local, and has a disconnected punctured spectrum.

(8)-(9) Let K be a field, and $R = K[s, t, u, v, x, y]/(sv^2x^2 - (s + t)vxuy + tu^2y^2)$. Let $\zeta_n = \left[\frac{sxy^n}{uv^n}\right] \in \check{H}^2(u, v; R) = \mathrm{H}^2_{(u,v)}(R)$.

- (a) Show that $(u, v, x, y) \in \mathrm{ann}_R(\zeta_n)$ for each n.
- (b) Let $S = K[s, t] \subseteq A = K[a = vx, b = uy, s, t] \subseteq R$. Show that A and R admit an $\mathbb{N}^3$-grading by $\deg(s) = \deg(t) = 0$, $\deg(u) = (1, 0, 1)$, $\deg(v) = (0, 1, 1)$, $\deg(x) = (1, 0, 0)$, $\deg(y) = (0, 1, 0)$.
- (c) Show that for $g \in S$, $g\zeta_n = 0$ if and only if $gab^n \in (a^{n+1}, b^{n+1})$.
- (d) Show that $(a^{n+1}, b^{n+1})A :_S ab^n = (s^n + s^{n-1}t + \cdots + st^{n-1} + t^n)$.
- (e) Compute the annihilator of ζ_n in R.
- (f) Show that $\mathrm{H}^2_{(u,v)}(R)$ has infinitely many associated primes.