Homework #3

Please write up and turn in at least *four* of the following problems at the beginning of class Monday, March 12.

(1) Show that $\operatorname{cd}(I_2(X_{2\times 4}), \mathbb{C}[X_{2\times 4}]) = 5$ and find a prime $\mathfrak{p}$ with $I_2(X_{2\times 4}) \subsetneq \mathfrak{p} \subset \mathbb{C}[X_{2\times 4}]$ and $\operatorname{cd}(\mathfrak{p}) = 4$.

(2) Show that if $(R, \mathfrak{m}, k)$ is local of dimension d and $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \neq \mathfrak{m}$, then $\mathrm{H}^i_\mathfrak{m}(R)$ has finite length for all $i < d$.

(3) Let $(R, \mathfrak{m}, k)$ be a regular local ring, and $\mathfrak{p} \in \operatorname{Spec}(R)$ of height $h \neq 0, \dim(R)$. Show that $\mathrm{H}^h_\mathfrak{p}(R)$ is neither artinian nor noetherian.

(4) This problem gives a proof that the invariant ring of SL_2 acting on $K[X_{2\times n}] = K\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$ is generated by the minors $\{\Delta_{ij}\}$ of X, if K has characteristic zero.

Define for $1 \leq i, j \leq n$ the *polarization operators* $E_{ij} := x_i \frac{\partial}{\partial x_j} + y_i \frac{\partial}{\partial y_j}$.

a) Show that each E_{ij} takes SL_2-invariants to SL_2-invariants.
b) Show that each E_{ij} sends the subalgebra $K[\{\Delta_{ij} \mid 1 \leq i < j \leq n\}]$ to itself.
c) Show that $K[X_{2\times n}]^{\mathrm{SL}_2}$ admits an $\mathbb{N}^n$-grading induced by the grading $|x_i| = |y_i| = \vec{e_i}$ on $K[X_{2\times n}]$.
d) Prove Cappelli's identity:

$$\begin{Vmatrix} E_{jj}+1 & E_{ij} \\ E_{ji} & E_{ii} \end{Vmatrix} = \begin{Vmatrix} x_i & x_j \\ y_i & y_j \end{Vmatrix} \circ \begin{Vmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{Vmatrix},$$

as differential operators on $K[X_{2\times n}]$, where $\| \star \|$ denotes determinant.

e) Prove that $K[X_{2\times n}]^{\mathrm{SL}_2} = K[\{\Delta_{ij} \mid 1 \leq i < j \leq n\}]$.

(5) This problem gives a proof[1] of the graded local duality theorem. Let $R = K[x_1, \ldots, x_d]$ be an $\mathbb{N}$-graded polynomial ring, with $\deg(x_i) = a_i$. Set $-a = a_1 + \cdots + a_d$.

For two graded R-modules, M and N, we set

$$\operatorname{Hom}_R(M, N)_i = \{\phi : M \to N \mid \phi \text{ is } R\text{-linear and } \phi(M_j) \subseteq N_{i+j} \text{ for all } j\ \},$$

and

$$\underline{\operatorname{Hom}}_R(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_R(M, N)_i.$$

If M and N are finitely generated R-modules, then $\operatorname{Hom}_R(M, N) = \underline{\operatorname{Hom}}_R(M, N)$ after forgetting the grading, and since M admits a graded free resolution by finitely generated modules, $\operatorname{Ext}^i_R(M, N)$ admits a natural grading. Similarly, $\operatorname{Tor}^R_i(M, N)$ admits a natural grading.

Define $(-)^*$ from graded R-modules to graded R-modules by the rule $M^* = \underline{\operatorname{Hom}}_K(M, K)$. From worksheet #2 we know that R^* is an injective hull for K, and from worksheet #3, we know that $\mathrm{H}^d_\mathfrak{m}(R) \cong R^*(-a)$ as graded modules.

a) Show that $M^* \cong \operatorname{Hom}_R(M, R^*)$ for any finitely generated graded R-module M.
b) Show that if M is finitely generated over R, then $M^{**} \cong M$.
c) Verify that $\mathrm{H}^i_\mathfrak{m}(M) \cong \operatorname{Tor}^R_{d-i}(M, R^*(-a))$.
d) State the graded versions of the Ext–Tor dualities.
e) Show that if M is a finitely generated graded R-module, then both dualities hold:

$$\mathrm{H}^i_\mathfrak{m}(M) \cong \operatorname{Ext}^{d-i}_R(M, R)^*(-a) \text{ and } \mathrm{H}^i_\mathfrak{m}(M)^* \cong \operatorname{Ext}^{d-i}_R(M, R)(a).$$

(6)–(8) Problems #3, #5, and #6 from the worksheet on Gorenstein rings.

[1] Note: The exact same arguments work for a graded ring R and integer a such that $\mathrm{H}^d_\mathfrak{m}(R) \cong R^*(-a)$.