Homework #2

Please write up and turn in at least *three* of the following problems at the beginning of class Monday, February 19. You are strongly encouraged to work out the rest of them, as well.

(1) Compute a k-basis for $\mathrm{H}^1_{(x)}(k[x,y])$ and show that this module is neither noetherian nor artinian.

(2) Let (V, pV, K) be a complete DVR with uniformizer $p \in \mathbb{Z}$. Let $R = V[\![x_1, \dots, x_t]\!]$. Note that R is local of dimension $t+1$ with residue field K.

a) Use the Čech complex to show that $\mathrm{H}^{t+1}_{(p,\underline{x})}(R) \cong \displaystyle\bigoplus_{\underline{\alpha}\,:\,\alpha_i<0} \frac{V[1/p]}{V} \cdot \underline{x}^{\underline{\alpha}}$.

b) Show that $\mathrm{Hom}_V^{(\underline{x})-\mathrm{cts}}(R, E_V(K)) \cong \mathrm{Hom}_V^{(\underline{x})-\mathrm{cts}}\left(R, \frac{V[1/p]}{V}\right)$ is an injective hull for K as an R-module. Use this description to show that $E_R(K) \cong \mathrm{H}^{t+1}_{(p,\underline{x})}(R)$.

(3) Let $(R, \mathfrak{m}, k)$ be a complete local ring.

a) Show that if $R \hookrightarrow M$ splits, then $R \otimes_R N \longrightarrow M \otimes_R N$ is injective for all R-modules N.

b) Show that if $R \otimes_R E_R(k) \longrightarrow M \otimes_R E_R(k)$ is injective, then $R \hookrightarrow M$ splits.

(Moral: $E_R(k)$ is the "least flat" module.)

(4) Problem #4 from worksheet #3.

(5) a) Give an example of a sequence of elements $f_1, \dots, f_t$ in a ring R and an integer i such that $\mathrm{H}^i(f_1^n, \dots, f_t^n; R) \neq 0$ for all n, but $\mathrm{H}^i_{(f_1,\dots,f_t)}(R) = 0$.

b) Give an example of an ideal I in a ring R, and R-module M, and an integer j such that $\mathrm{Ext}^j_R(R/I^n, M) \neq 0$ for all n, but $\mathrm{H}^j_I(M) = 0$.

(6) Show that if R is a regular ring of dimension d, the minimal injective resolution of R is of the form

$$0 \left(\to R\right) \to E_R(R) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} E_R(R/\mathfrak{p}) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=2} E_R(R/\mathfrak{p}) \to \cdots \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=d} E_R(R/\mathfrak{p}) \to 0.$$

(7) Use the previous problem to show that if R is regular and $\mathrm{ht}(I) = h$, then

$$\mathrm{Ass}\left(\mathrm{H}^h_I(R)\right) = \{\mathfrak{p} \in \mathrm{Min}(I) \mid \mathrm{ht}\,\mathfrak{p} = h\}.$$