Homework #1

Please write up and turn in at least *four* of the following problems at the beginning of class Monday, February 5. You are strongly encouraged to work out the rest of them, as well.

(1) Let k be a field. Find a free resolution of $M = k[x,y]/(y)$ over $R = k[x,y]/(xy)$. Use it to compute $\operatorname{Ext}^i_R(M,k)$ for all $i \geq 0$, where we interpret k as a module via $k \cong R/(x,y)$.

(2) Use the definition of Cohen-Macaulay to show that

$$R = \frac{k\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}_{\mathfrak{m}}}{(uy - vx, uz - wx, vz - wy)}$$

is Cohen-Macaulay, where $\mathfrak{m}$ is the ideal generated by the (images of the) variables.[1]

(3) Find a free resolution of the cyclic module $M = R/(uy - vx, uz - wx, vz - wy)$ over

$$R = k\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}_{\mathfrak{m}}.$$

Compute $\operatorname{Ext}^i_R(M,R)$ for all $i \geq 0$.

One possibility to find a free resolution is to follow the following steps:

- Compute $\det \begin{bmatrix} u & v & w \\ u & v & w \\ x & y & z \end{bmatrix}$ by expanding the first row.
- Use #2 and Auslander-Buchsbaum to determine the projective dimension of M, and determine the rank of the remaining free modules.

(4) Let R be a local ring. A f.g. module is called *maximal Cohen-Macaulay* or *MCM* if $\operatorname{depth} M = \dim R$. Show that, if R has an MCM module, R is regular if and only if every f.g. MCM module over R is free.

(5) Compute the minimal injective resolution of $\mathbb{C}[x]$ as a $\mathbb{C}[x]$-module. Write each injective as a direct sum of indecomposable injectives. Use this to compute $\mathrm{H}^i_{(x)}(\mathbb{C}[x])$.

(6) Let R be a Noetherian ring, I an ideal of R, and E an injective R-module. Show that $\Gamma_I(E)$ is injective, and compute its direct sum decomposition into indecomposables in terms I and the direct sum decomposition of E.

(7) Let $(A,\nu) \to (R,\mathfrak{m})$ be a local homomorphism[2] of local rings. Assume that $A/\nu \cong R/\mathfrak{m}$. Suppose that there exists an ideal J of R such that the composed map $A \to R \to R/J$ is an isomorphism.[3] Show that $E_R(R/\mathfrak{m}) \cong \operatorname{Hom}_A^{J\text{-cts}}(R, E_A(A/\nu))$. Use this to give an explicit description of $E_R(R/\mathfrak{m})$ when $R = \mathbb{Z}_p[\![\underline{x}]\!]$, where $\mathbb{Z}_p$ denotes the p-adic integers.

(8) Problem #1 from the worksheet on Koszul homology and CM rings

(9) Problem #3 from the worksheet on Matlis duality with coefficient fields

[1] If you get stuck at step 1, assume that R is a domain. Of course, it's better to prove R is a domain.
[2] This means that the image of ν is contained in $\mathfrak{m}$.
[3] We call A a *ring retract* of R in this setting.