Math 614, Fall 2018, Homework #4

Some basic M2 commands:

- `ideal( )` turns a list of elements into the ideal they generate.
- If `R` is a ring and `I` is an ideal, form the quotient ring by `R/I`.
- If `R` is a quotient of a polynomial ring with generators `x_1`, ..., `x_n`, and `S` is another ring, then `map(S,R,{a_1,..., a_n })` is the homomorphism from `R` to `S` sending the image of `x_i` to `a_i`.
- `ker( )` returns the kernel of a map.
- `primaryDecomposition( )` of an ideal gives a primary decomposition.
- `ass( )` of an ideal returns the set of associated primes of `R/I`.
- `I : f` computes the colon ideal $I : f$.

Use Macaulay2 to help solve the following problems.

(1) Find presentations[1] for the following $\mathbb{Q}$-algebras.
  (a) $R = \mathbb{Q}[x^3, x^4, x^5] \subseteq \mathbb{Q}[x]$.
  (b) $S = \mathbb{Q}[x^3, x^4, x^5, y, xy] \subseteq \mathbb{Q}[x, y]$.
  (c) $T = \mathbb{Q}[x^2, x^3 - x, y, xy] \subseteq \mathbb{Q}[x, y]$.
  (d) $U = \mathbb{Q}\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \dfrac{\mathbb{Q}[u, v, x, y, z]}{(x^3 + y^3 + z^3)}$.

(2) Find the associated primes and primary decompositions of the following ideals.
  (a) $I = (u^2 - x^2, v^2 - y^2, uv - xy) \subseteq \mathbb{Q}[x, y, u, v]$.
  (b) The *square* of the ideal $I_2(X_{3\times 3})$ of 2×2 minors of a 3×3 matrix of indeterminates.
  (c) The ideal (ux, uy) in the ring U above.

(3) For the ideal $\mathfrak{p}$ you found in #1(a), find elements in $\mathfrak{p}^{(2)} \setminus \mathfrak{p}^2$ and in $\mathfrak{p}^{(3)} \setminus \mathfrak{p}^3$.

(4) For each of the following homomorphisms and primes in the source, find a prime that contracts to it, or else show that no prime contracts to it.
  (a) The inclusion map

$$\mathbb{Q}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}] \subseteq \mathbb{Q}[X_{2\times 4}]$$

  where $X_{2\times 4}$ is a 2×4 matrix of indeterminates, and Δ_{ij} is the 2×2 minor coming from columns i and j, with the prime $\mathfrak{p} = (\Delta_{12}, \Delta_{13}, \Delta_{14})$. Check that $\mathfrak{p}$ is prime!
  (b) The inclusion map $\mathbb{Q}[x, y, z] \subseteq \dfrac{\mathbb{Q}[x, y, z, u, v, w]}{(z^2 + yv^2 + w^2, u^2x + w^2 + 4u^2y)}$, with the primes $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{p}_3 = (y, z)$.

[1] That is, express each of the following as a quotient of a polynomial ring by an explicit ideal.