Homework #3 volunteered solutions

Problem 1. *Show that* $\operatorname{cd}(I_2(X_{2\times 4}), \mathbb{C}[X_{2\times 4}]) = 5$ *and find a prime* $\mathfrak{p}$ *with* $I_2(X_{2\times 4}) \subsetneq \mathfrak{p} \subset \mathbb{C}[X_{2\times 4}]$ *and* $\operatorname{cd}(\mathfrak{p}) = 4$.

Solution 1.

Lemma 1. *Let* $R \subseteq S$ *be an inclusion of noetherian rings that splits as* R*-modules, and* $I \subseteq R$ *be an ideal. Then* $\operatorname{cd}(I, R) = \operatorname{cd}(IS, S)$

Proof. We have that $\operatorname{cd}(I, R) \geq \operatorname{cd}(I, S) = \operatorname{cd}(IS, S)$ in general. Since R is a direct summand of S, $\mathrm{H}^i_I(R)$ is a direct summand of $\mathrm{H}^i_I(S) \cong \mathrm{H}^i_{IS}(S)$ for all i. This gives the other inequality. □

We know that $\mathbb{C}[X_{2\times 4}]^{\mathrm{SL}_2(\mathbb{C})} = \mathbb{C}[\{\Delta_{ij} \mid 1 \leq i < j \leq 4\}]$ is a direct summand of $\mathbb{C}[X_{2\times 4}]$. From this lemma, we know that $\operatorname{cd}(I_2(X_{2\times 4}), \mathbb{C}[X_{2\times 4}]) = \operatorname{cd}((\{\Delta_{ij}\}), \mathbb{C}[\{\Delta_{ij}\}])$, and since $(\{\Delta_{ij}\})$ is a maximal ideal, this cohomological dimension is just the dimension of the f.g. domain $\mathbb{C}[\{\Delta_{ij}\}]$.

The dimension is at least 5, since, after specializing $x_{11} \mapsto 1$, $x_{12} \mapsto 0$, $x_{21} \mapsto 0$, the minors become

$$x_{22}, x_{23}, x_{24}, x_{13}x_{22}, x_{14}x_{22}, \Delta_{34},$$

and the first five are obviously algebraically independent.

The dimension is at most 5 since the minors satisfy the relation $\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}$.

Finally, note that $\mathfrak{p} = (x_{11}, x_{12}, x_{13}, x_{14}) \supsetneq I_2(X_{2\times 4})$, and $\operatorname{cd}(\mathfrak{p}) = 4$.

Problem 2. *Show that if* $(R, \mathfrak{m}, k)$ *is local of dimension* d *and* $R_{\mathfrak{p}}$ *is Cohen-Macaulay for all* $\mathfrak{p} \neq \mathfrak{m}$, *then* $\mathrm{H}^i_{\mathfrak{m}}(R)$ *has finite length for all* $i < d$.

Solution 2.

Problem 3. *Let* $(R, \mathfrak{m}, k)$ *be a regular local ring, and* $\mathfrak{p} \in \operatorname{Spec}(R)$ *of height* $h \neq 0, \dim(R)$. *Show that* $\mathrm{H}^h_{\mathfrak{p}}(R)$ *is neither artinian nor noetherian.*

Solution 3 (Eamon Quinlian). For simplicity denote $H = H^h_P(R)$. Let us first show that H is not finitely generated (i.e. noetherian). If it was then $H_P = H^h_{PR_P}(R_P)$ would be finitely generated over R_P. Observe that since R is regular so is R_P, and that since $P \neq 0$ R_P has positive dimension. We thus reduce to the following claim.

Claim.- Let (S, n, L) be a regular local ring of dimension $d > 0$. Then $H^d_n(S)$ is not finitely generated.

Proof of Claim: We may assume S is complete. Recall that because S is regular $H^d_n(S) = E_S(L)$ and hence, by completeness, $H^d_n(S)^\vee = S$. Since S is positive-dimensional, S is not artinian. Therefore, $H^d_n(S) = S^\vee$ is not finitely-generated. □

We now show that H is not artinian. First observe that as $H_P \neq 0$ by the above considerations, $H \neq 0$. Thus we may pick some $0 \neq \alpha \in H$ and denote by N the submodule generated by α. It suffices to show that N is not artinian.

Since $\operatorname{Ass}(H) = \{P\}$ (HW2 #7), we have that $\operatorname{Ass}(N) = \{P\}$. As N is finitely-generated all elements of $m \setminus P$ are nonzerodivisors. As $P \neq m$ we may pick one such $y \in m \setminus P$. The following claim concludes the exercise.

Claim.- The chain $N \supsetneq yN \supsetneq y^2N \supsetneq \cdots$ does not stabilize.

Proof of Claim: As y is a nonzerodivisor, if $y^nN = y^{n+1}N$ for some n then $N = yN$. As $y \in m$ this would imply $N = mN$, thus $N = 0$ by Nakayama's lemma – a contradiction.

Problem 4. *This problem gives a proof that the invariant ring of* SL_2 *acting on* $K[X_{2\times n}] = K\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$ *is generated by the minors* $\{\Delta_{ij}\}$ *of* X, *if* K *has characteristic zero.*

Define for $1 \leq i, j \leq n$ *the* polarization operators $E_{ij} := x_i \frac{\partial}{\partial x_j} + y_i \frac{\partial}{\partial y_j}$.

(1) Show that each E_{ij} *takes* SL_2*-invariants to* SL_2*-invariants.*

(2) Show that each E_{ij} sends the subalgebra $K[\{\Delta_{ij} \mid 1 \le i < j \le n\}]$ to itself.
(3) Show that $K[X_{2\times n}]^{\mathrm{SL}_2}$ admits an $\mathbb{N}^n$-grading induced by the grading $|x_i| = |y_i| = \vec{e_i}$ on $K[X_{2\times n}]$.
(4) Prove Cappelli's identity:

$$\begin{Vmatrix} E_{jj}+1 & E_{ij} \\ E_{ji} & E_{ii} \end{Vmatrix} = \begin{Vmatrix} x_i & x_j \\ y_i & y_j \end{Vmatrix} \circ \begin{Vmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{Vmatrix},$$

as differential operators on $K[X_{2\times n}]$, where $\| \star \|$ denotes determinant.
(5) Prove that $K[X_{2\times n}]^{\mathrm{SL}_2} = K[\{\Delta_{ij} \mid 1 \le i < j \le n\}]$.

Solution 4 (Zhan Jiang). (1) Let $f(\underline{x}, \underline{y})$ be an SL_2-invariant element. Then we want to show that $E_{ij}(f)$ is also SL_2-invariant. Let $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element in SL_2 and write $x'_i = ax_i + by_i$ and $y'_i = cx_i + dy_i$. We also write $\partial^x_i f := \frac{\partial f}{\partial x_i}$ and $\partial^y_i f := \frac{\partial f}{\partial y_i}$ for $1 \le i \le n$.

On one hand, we have

$$\begin{aligned} \phi(E_{ij}(f)) &= \phi(x_i\partial^x_j f + y_i\partial^y_j f) \\ &= x'_i\partial^x_j f(\underline{x}', \underline{y}') + y'_i\partial^y_j f(\underline{x}', \underline{y}') \end{aligned}$$

On the other hand, we have

$$\begin{aligned} E_{ij}(\phi(f)) &= x_i\partial^x_j\phi(f) + y_i\partial^y_j\phi(f) \\ &= (ax_i\partial^x_j f(\underline{x}', \underline{y}') + cx_i\partial^y_j f(\underline{x}', \underline{y}')) + (by_i\partial^x_j f(\underline{x}', \underline{y}') + dy_i\partial^y_j f(\underline{x}', \underline{y}')) \\ &= (ax_i + by_i)\partial^x_j f(\underline{x}', \underline{y}') + (cx_i + dy_i)\partial^y_j f(\underline{x}', \underline{y}') \\ &= x'_i\partial^x_j f(\underline{x}', \underline{y}') + y'_i\partial^y_j f(\underline{x}', \underline{y}') \end{aligned}$$

Hence $\phi(E_{ij}(f)) = E_{ij}(\phi(f))$. So if $\phi(f) = f$, then $\phi(E_{ij}(f)) = E_{ij}(f)$.

(2) It's quite easy to see that E_{ij} is a differential operator, hence if it maps each generator back to the subalgebra, then it sends everything in the subalgebra to itself.

Write $\Delta_{ij} = x_iy_j - x_jy_i$, and apply E_{ik} we have

$$\begin{aligned} E_{ij}(\Delta_{jk}) &= \Delta_{ik} \\ E_{ij}(\Delta_{kl}) &= 0 \end{aligned}$$

So we're done.

(3) For any $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in SL_2, since a, b cannot be both zero. The action $x_i \mapsto ax_i + by_i$ is homogeneous and it's the same for $y_i \mapsto cx_i + dy_i$. So if f is SL_2 invariant, then each homogeneous part has to be invariant. So $K\left[X_{2\times n}\right]^{\mathrm{SL}_2}$ admints the grading.

(4) We have to prove

$$(E_{jj} + 1) \circ E_{ii} - E_{ij} \circ E_{ji} = (x_iy_j - x_jy_i)(\partial^x_i\partial^y_j - \partial^x_j\partial^y_i)$$

By using the relations that all operators are commutative except that $[\partial^x_i, x_i] = 1$ and $[\partial^y_i, y_i] = 1$.

$$\begin{aligned}(E_{jj}+1)\circ E_{ii}-E_{ij}\circ E_{ji}=&(x_j\partial_j^x+y_j\partial_j^y+1)\circ(x_i\partial_i^x+y_i\partial_i^y)-(x_i\partial_j^x+y_i\partial_j^y)\circ(x_j\partial_i^x+y_j\partial_i^y)\\=&x_j\partial_j^x x_i\partial_i^x+x_j\partial_j^x y_i\partial_i^y+y_j\partial_j^y x_i\partial_i^x+y_j\partial_j^y y_i\partial_i^y+x_i\partial_i^x+y_i\partial_i^y\\&-x_i\partial_j^x x_j\partial_i^x-x_i\partial_j^x y_j\partial_i^y-y_i\partial_j^y x_j\partial_i^x-y_i\partial_j^y y_j\partial_i^y\\=&x_jx_i\partial_j^x\partial_i^x+x_jy_i\partial_j^x\partial_i^y+y_jx_i\partial_j^y\partial_i^x+y_jy_i\partial_j^y\partial_i^y+x_i\partial_i^x+y_i\partial_i^y\\&-x_i\partial_i^x-x_ix_j\partial_j^x\partial_i^x-x_iy_j\partial_j^x\partial_i^y-y_ix_j\partial_j^y\partial_i^x-y_i\partial_i^y-y_iy_j\partial_j^y\partial_i^y\\=&x_jy_i\partial_j^x\partial_i^y+y_jx_i\partial_j^y\partial_i^x-x_iy_j\partial_j^x\partial_i^y-y_ix_j\partial_j^y\partial_i^x\\=&(x_jy_i-x_iy_j)\partial_j^x\partial_i^y+(y_jx_i-y_ix_j)\partial_j^y\partial_i^x\\=&(x_iy_j-x_jy_i)(\partial_i^x\partial_j^y-\partial_j^x\partial_i^y)\end{aligned}$$

(5) For notational reason write $R=K\left[\left\{\Delta_{ij}\middle|1\le i<j\le n\right\}\right]$ and $S=K\left[X_{2\times n}\right]^{\mathrm{SL}_2}$. Since both are graded rings, we only need to show that all homogeneous elements in S are in R. We first prove following lemma:

Lemma: For any nonzero homogeneous element $f\in S$, the degree of f cannot be $k\cdot\overrightarrow{e}_i$ for any natural number k. In other words, f cannot be an element in $K[x_i,y_i]$.

Proof. Suppose $f\in K\left[x_i,y_i\right]$, then we can write

$$f=c_0x_i^k+c_1x_i^{k-1}y_i+\cdots+c_ky_i^k$$

where $c_j\in K$. After apply $\begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix}\in\mathrm{SL}_2$ to f we get

$$\begin{aligned}c_0(ax_i)^k+c_1(ax_i)^{k-1}(y_i/a)+\cdots+c_k(y_i/a)^k=&c_0x_i^k+c_1x_i^{k-1}y_i+\cdots+c_ky_i^k\\ \Leftrightarrow c_j=&c_ja^{k-2j}\end{aligned}$$

for any nonzero $a\in K$. Since K has char 0 and hence it has infinitely many elements. We conclude that $c_j=0$ for all j. But then f is zero. □

Now we are ready to prove $R=S$. If a homogeneous element $f\in S$ is of degree $a_1\overrightarrow{e}_1+a_n\overrightarrow{e}_n$, call $a_1+\cdots+a_n$ the total degree of f. Choose a homogeneous element $f\in S\setminus R$ with smallest possible total degree. Assume WLOG that $\deg(f)=a_1\overrightarrow{e}_1+\cdots+a_l\overrightarrow{e}_l$ for some $1\le l\le h$ with all $a_i>0$.

If f involves x_j or y_j, then easy calculation shows that $\deg(E_{ij}(f))=\deg(f)+\overrightarrow{e}_i-\overrightarrow{e}_j$ if $E_{ij}(f)\neq 0$. Hence by appyling E_{1i} $(1<i\le l)$ a_i times consectively for each i we can find an element of homogeneous degree $k\overrightarrow{e}_1$, which is impossible by Lemma. So there is some intermediate step f_1 such that $f_1\neq 0$ but $E_{1i}(f_1)=0\in R$. This tells us that we can find an intermediate step f_2 such that $f_2\notin R$ but $E_{1j}(f_2)\in R$. Now apply Cappelli's operator

$$\begin{vmatrix}E_{11}+1 & E_{j1}\\ E_{1j} & E_{jj}\end{vmatrix}=\Delta_{j1}\circ D$$

to f_2. We have

$$(E_{11}+1)E_{jj}(f_2)-E_{j1}E_{1j}(f_2)=\Delta_{j1}(D(f_2))$$

Note that $D(f_2)$ has smaller total degree, so it must be in R, therefore RHS is in R. Since $E_{1j}(f_2)\in R$ by our choice, so is $E_{j1}E_{1j}(f_2)$. Hence we have $(E_{11}+1)E_{jj}(f_2)\in R$.

But if we write f_2 as

$$f_2=f_hx_j^h+f_{h-1}x_j^{h-1}y_j+\cdots+f_0y_j^h$$

where $f_h,f_{h-1},...$ doesn't involve x_j or y_j. Then easy calculation shows that

$$E_{jj}(f_2)=hf_2$$

Hence the operator $(E_{11}+1)E_{jj}$ acts as a nonzero scalar on f_2, which implies that $f_2\in R$, a contradiction! So there is no such $f\in S\setminus R$ for us to start. Hence $R=S$.

Problem 5. *This problem gives a proof of the graded local duality theorem. Let $R = K[x_1, \dots, x_d]$ be an $\mathbb{N}$-graded polynomial ring, with $\deg(x_i) = a_i$. Set $-a = a_1 + \cdots + a_d$. From worksheet #2 we know that R^* is an injective hull for K, and from worksheet #3, we know that $\mathrm{H}^d_{\mathfrak{m}}(R) \cong R^*(-a)$ as graded modules.*

(1) Show that $M^* \cong \mathrm{Hom}_R(M, R^*)$ for any finitely generated graded R-module M.
(2) Show that if M is finitely generated over R, then $M^{**} \cong M$.
(3) Verify that $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{Tor}^R_{d-i}(M, R^*(-a))$.
(4) State the graded versions of the Ext–Tor *dualities.*
(5) Show that if M is a finitely generated graded R-module, then both dualities hold:

$$\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{Ext}^{d-i}_R(M, R)^*(-a) \text{ and } \mathrm{H}^i_{\mathfrak{m}}(M)^* \cong \mathrm{Ext}^{d-i}_R(M, R)(a).$$

Solution 5 (Zhan Jiang). (1) If we forget grading, then

$$\mathrm{Hom}_R(M, \mathrm{Hom}_K(R, K)) \cong \mathrm{Hom}_K(M \otimes_R R, K) \cong \mathrm{Hom}_K(M, K)$$

So the only thing remains to show is that this isomorphism is degree-preserving.

Let $\alpha \in M^*$ be a homogeneous element of degree i, i.e. $\alpha(M_j) \subseteq J_{i+j}$. The identification above is given by

$$\alpha \mapsto (u \mapsto \alpha_u)$$

where $\alpha_u : r \mapsto \alpha(ru)$. For any homogeneous element $u \in M$ of degree k, we want to determine the degree of the map α_u. For any homogeneous element $r \in R$ of degree l, $\alpha_u(r) = \alpha(ru) \in K_{k+l+i}$. So $\alpha_u \in \mathrm{Hom}_K(R, K)_{k+i}$. So the map $u \mapsto \alpha_u$ is of degree i. So this is a degree-preserving isomorphism.

(2) There is a natural map

$$M \to \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_K(M, K), K)$$
$$u \mapsto e_u\,(\phi \mapsto \phi(u))$$

We check that this preserves degree: let $u \in M$ be a homogeneous element of degree k, then we want to show that e_u is of degree k, i.e. $e_u(\mathrm{Hom}_K(M, K)_i) \subseteq K_{i+k}$. Let ϕ be a homogeneous element in $\mathrm{Hom}_K(M, K)_i$, since $u \in M_k$, we have $\phi(u) \in K_{k+i}$. Hence $e_u(\mathrm{Hom}_K(M, K)_i \subseteq K_{k+i}$. So it's of degree k.

Notice that the only nonzero part of K is K_0, so e_u maps $\mathrm{Hom}_K(M, K)_{-k} = \mathrm{Hom}_K(M_k, K)$ to K and other degrees to zero. Hence $e_u \in \mathrm{Hom}_K(\mathrm{Hom}_K(M_k, K), K)$. So the natural map above restricts to a K-linear map $f_k : M_k \to \mathrm{Hom}_K(\mathrm{Hom}_K(M_k, K), K)$. Note that if M is finitely generated, then each M_k is a finite-dimensional K-vector space. The natural map f_k is an isomorphism. So the natural map at the beginning is an isomorphism.

(3) The Čech complex is a graded flat resolution of $R^*(-a)$. Since both sides are obtained by tensoring with M and taking (co)homology, they are literally the same graded module.

(4) Under the assumption above, for any finitely generated graded R-modules M and N,
(a) $\mathrm{Tor}^R_i(M, N)^* \cong \mathrm{Ext}^i_R(M, N^*)$
(b) $\mathrm{Tor}^R_i(M, N^*) \cong \mathrm{Ext}^i_R(M, N)^*$

(5) By duality above, we have $\mathrm{Tor}^R_i(M, R^*) \cong \mathrm{Ext}^i_R(M, R)^*$ so then $\mathrm{Tor}^R_i(M, R^*)(-a) \cong \mathrm{Ext}^i_R(M, R)^*(-a)$. Since shifting degree commutes with tensor and taking (co)homology, we have $\mathrm{Tor}^R_i(M, R^*)(-a) \cong \mathrm{Tor}^R_i(M, R^*(-a)) \cong H^{d-i}_m(M)$. Combine all these we have

$$H^i_m(M) \cong \mathrm{Ext}^{d-i}_R(M, R)^*(-a)$$

Apply graded dual $(-)^*$ we have

$$H^i_m(M)^* \cong (\mathrm{Ext}^{d-i}_R(M, R)^*(-a))^* = \mathrm{Ext}^{d-i}_R(M, R)^{**}(a) \cong \mathrm{Ext}^{d-i}_R(M, R)(a)$$

Problem 6. *Show that if R is a Gorenstein local ring, and M is a finitely generated R-module, then M has finite projective dimension if and only if M has finite injective dimension.*

Solution 6. First we observe the following fact: if

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of (not necessarily f.g.) modules, and $\mathrm{Tor}^R_{\gg 0}(k, -) = 0$ for two of the three modules, then likewise for the third. Similarly, if $\mathrm{Ext}_R^{\gg 0}(k, -) = 0$ for two of the three modules, then likewise for the third. Both of these facts follow immediately from the long exact sequence.

Let M be a f.g. module of finite projective dimension. Then M has a finite free resolution. We claim that $\mathrm{Ext}_R^{\gg 0}(k, M) = 0$. We induce on the projective dimension of M. Indeed, if M is free, then the injective dimension of $M \cong R^{\oplus a}$ is finite, by the Gorenstein hypothesis. Otherwise, take a SES

$$0 \to L \to F \to M \to 0$$

with F free; the projective dimension of L is smaller than that of M. By the observation above, and the induction hypothesis applied to L, the claim follows. Then, since $\mathrm{Ext}_R^{\gg 0}(k, M) = 0$ and M is finitely generated, the injective dimension of M is finite.

Now, let M be a f.g. module of finite injective dimension, so M has a finite injective resolution. We claim that $\mathrm{Tor}^R_{\gg 0}(k, M) = 0$. We induce on the injective dimension of M. To deal with the base case, it suffices to show that $\mathrm{Tor}^R_{>\dim(R_\mathfrak{p})}(k, E_R(R/\mathfrak{p})) = 0$ for any prime $\mathfrak{p}$. Since $R_\mathfrak{p}$ is Gorenstein, $E_R(R/\mathfrak{p}) \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong \mathrm{H}^{\dim(R_\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$, which has a resolution of length $\dim(R_\mathfrak{p})$ by flat modules, namely, $\check{C}^\bullet(\underline{f}; R_\mathfrak{p})$ for a SOP $\underline{f}$ of $R_\mathfrak{p}$. Since Tor can be computed by flat resolutions, this takes care of the base case. Otherwise, take a SES

$$0 \to M \to E_R(M) \to N \to 0;$$

the injective dimension of N is smaller than that of M. By the observation above and the inductive hypothesis, the inductive step, and hence also the claim, follows. Then, since $\mathrm{Tor}^R_{\gg 0}(k, M) = 0$ and M is finitely generated, the projective dimension of M is finite.