Math 614, Fall 2018, Homework #3

Please write up and turn in at least *four* of the following problems at the beginning of class Thursday, October 18. You are strongly encouraged to work out the rest of them, as well.

(1) Let $\mathbb{Z}[x] \xrightarrow{\varphi} \dfrac{\mathbb{Z}[x,y]}{(2x+3y)}$ be the inclusion map. Let φ^* be the induced map on spectra.
  (a) Describe the sets $(\varphi^*)^{-1}((2,x))$, $(\varphi^*)^{-1}((x))$, and $(\varphi^*)^{-1}((2))$.
  (b) What is the fiber ring over each of the primes above?

(2) (a) Let φ be a ring homomorphism. Show that the closure of the image of the induced map on spectra is $V(\ker(\varphi))$.[1]
  (b) Let K be a field, and $K[x,y] \xrightarrow{\varphi} K[x,y]$ be given by the rule $\varphi(x) = x, \varphi(y) = xy$, and $\varphi|_K = \mathrm{id}_K$. Show that the image of φ^* is neither open nor closed.

(3) If R is a domain, F its fraction field, and M is an R-module, the *rank* of M is the F-vector space dimension of $F \otimes_R M \cong M_{(0)}$.
  (a) Show that M has rank zero if and only if it is a torsion module[2].
  (b) Show that M injects into $M_{(0)}$ if and only if M is torsionfree[3].
  (c) Show that if M is finitely generated, then the rank of M is finite.
  (d) Show that if M is finitely presented and torsion free, then there are short exact sequences of the form

$$0 \to M \to R^{\oplus \mathrm{rank(M)}} \to T \to 0 \qquad \text{and} \qquad 0 \to R^{\oplus \mathrm{rank(M)}} \to M \to T' \to 0$$

  where T and T' are torsion modules.

(4) An R-module E is *injective* if the functor $\mathrm{Hom}_R(-, E)$ turns short exact sequences into short exact sequences.
  (a) Show that E is injective if and only if, for any injective map $M \xrightarrow{\alpha} N$, and any homomorphism $M \xrightarrow{\beta} E$, there is a map $N \xrightarrow{\gamma} E$ such that $\gamma \circ \alpha = \beta$.
  (b) Show that if R is a domain, E is an injective module, then for every $e \in E$ and $r \in R$, there is some $e' \in E$ such that $re' = e$.
  (c) Show that if R is a local domain that is not a field and E is a nonzero injective R-module, then E is *not* finitely generated.

(5) (a) Baer's criterion says that an R-module is injective if and only if, for every ideal $I \subseteq R$, the restriction map $\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(I, E)$ is surjective. Either prove this yourself or look up the proof; you don't need to write anything for this part.
  (b) Let R be a Noetherian ring. Show that if E is injective, then $E_{\mathfrak{p}}$ is an injective $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
  (c) Show that if R is a Noetherian domain that is not a field, and E is a nonzero injective module, then E is *not* finitely generated.

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[1] Hint: Quotient out by $\ker(\varphi)$ and localize at a minimal prime.
[2] Recall that this means that every element of M is killed by some nonzero element of R.
[3] Recall that this means that no element of M is killed by some nonzero element of R.

(6) In this problem, we we will show the spectrum of a ring is disconnected if and only if the ring can be written as a nontrivial direct product.
  (a) Show that R is a direct product of two nonzero rings if and only if there is a nonzero idempotent element in R.
  (b) Show that if $e^2 = e \neq 0$, then $\mathrm{Spec}(R)$ is the disjoint union of $V((e))$ and $V((1-e))$.
  (c) Show that if $\mathrm{Spec}(R)$ is disconnected, then there are ideals I, J such that $I+J = R$ and $IJ \subseteq \sqrt{(0)}$.
  (d) Conclude the proof of the statement.[4]

(7) Problem 5 (a) and (b) from the Gröbner worksheet.

[4]Hint: Take $i \in I, j \in J$ with $i + j = 1$ and $(ij)^n = 0$ from the previous part, expand $1 = (i + j)^{2n-1}$, and separate into two suitable parts.