Math 614, Fall 2018, Homework #2

Please write up and turn in at least *four* of the following problems at the beginning of class Thursday, October 4. You are strongly encouraged to work out the rest of them, as well.

(1) Open up the web interface for Macaulay2 (it's easy to find on google; otherwise there is a link from the course website). Define a polynomial ring in variables $\{x_{ij}\}_{i,j=1,2,3}$ over $\mathbb{Q}$, the ideal $I = (x_{11}^2 + x_{12}x_{21}, x_{11}x_{21} + x_{21}x_{22}, x_{11}x_{12} + x_{12}x_{22}, x_{12}x_{21} + x_{22}^2)$, and compute its radical as follows:

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R=QQ[x11,x12,x13,x21,x22,x23,x31,x32,x33]
I=ideal(x11^2+x12*x21,x11*x21+x21*x22,x11*x12+x12*x22,x12*x21+x22^2)
radical(I)
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or, you could define the same I by

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M=matrix{{x11,x12},{x21,x22}}
N=M^2
I=ideal(N)
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Now use Macaulay2 to compute the radical of the ideal generated by the entries of M^3, where $M = [x_{ij}]_{i,j=1,2,3}$.

(2) Let A be a finitely generated $\mathbb{Z}$-algebra. Show that for any maximal ideal $\mathfrak{m}$ of A, the residue field $A/\mathfrak{m}$ is finite.

(3) Consider a system of polynomial equations and inequations in $\mathbb{C}[x_1, \dots, x_n]$:

$$(\clubsuit) \qquad f_1(\underline{x}) = \cdots = f_a(\underline{x}) = 0, \quad g_1(\underline{x}) \neq 0 \ , \ \dots \ , \ g_b(\underline{x}) \neq 0.$$

Let $A = \mathbb{Z}[\text{all coefficients of } f_1, \dots, f_a, g_1, \dots, g_b] \subseteq \mathbb{C}$; tautologically, we can interpret $(\clubsuit)$ as a system of polynomial equations and inequations with coefficients in A.

(a) Show that if $(\clubsuit)$ has a solution over $\mathbb{C}$, then $(\clubsuit)$ has a solution over some A-algebra that is a finite field $\mathbb{F}_q$.

(b) Show that if $(\clubsuit)$ has no solution over $\mathbb{C}$, then there is some A-algebra that is a finite field $\mathbb{F}_q$ over which $(\clubsuit)$ has no solution.

(4) Let K be a field, and $\Phi_{\underline{f}} : K^n \to K^n$ be given by the rule

$$\Phi_{\underline{f}}(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$$

where $\underline{f} = f_1, \dots, f_n$ are polynomials in $x_1, \dots, x_n$.

(a) Show that $\Phi_{\underline{f}}$ is not surjective if and only if there is some $\underline{a} \in K^n$ such that the system

$$f_1(\underline{x}) = a_1, \dots, f_n(\underline{x}) = a_n$$

has no solution over K.

(b) Express the condition for $\Phi_{\underline{f}}$ to be injective in terms of some system of equations and inequations having no solution over K.
(c) Prove the Ax-Grothendieck Theorem: If $K = \mathbb{C}$, and $\Phi_{\underline{f}}$ as above is injective, then it is surjective.

(5) Let R be a ring of characteristic $p > 0$.
(a) Show that the map $F : R \to R$ given by $F(r) = r^p$ is a ring homomorphism.
(b) Show that F is module-finite if and only if it is algebra-finite.
(c) Show that the map on spectra induced by F is the identity map.

(6) Let R be a finitely generated algebra over a field. Let $\mathfrak{p}$ be a prime ideal. Show that $\mathfrak{p} = \bigcap_{\mathfrak{m} \text{ maximal}, \mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}$.

A ring with this property is called a *Jacobson ring*, or a *Hilbert ring*. If you want a harder version of this problem, show that if R is Jacboson, then any finitely generated R-algebra is Jacobson.

(7) Let K be an algebraically closed field, $R = K[x_1, \dots, x_n]$ be a polynomial ring, and G be a finite group with a representation on $V = K^n$. This induces a degree-preserving action of G on R by $g \cdot f = f \circ g^{-1}$ (as functions from K^n to K).
(a) Show that if $G \cdot \underline{v} \neq G \cdot \underline{w}$, there there exists $f(\underline{x}) \in R^G$ such that $f(\underline{v}) \neq f(\underline{w})$.
(b) Show that there is a commutative diagram of the form

$$\begin{array}{ccc} V & \xrightarrow{\cong} & \mathrm{Max}(R) \\ \downarrow & & \downarrow \\ V/G & \xrightarrow{\cong} & \mathrm{Max}(R^G), \end{array}$$

where V/G is the set of G-orbits of V, and the second vertical map is induced by the inclusion.