Math 614, Fall 2018, Homework #1 Comments

- For problem #2, we can't assume that if an element in a domain is not irreducible, then it is divisible by an irreducible element.

  Here is a counterexample: let $R = \bigcup_{n\in\mathbb{N}} \mathbb{C}[\![x^{1/n}]\!]$. You can check that every element of R can be written as a product $r = x^{m/n}u$ for some $m/n \in \mathbb{Q}$ and u a unit; this follows pretty easily from the analogous statement in $\mathbb{C}[\![x]\!]$. From this decomposition, we find that there are no irreducible elements in R at all! Indeed, if r as above is not a unit, then we can write $r = (x^{m/2n}u)(x^{m/2n})$, so any nonunit is a product of two other nonunits.

  For the sake of the problem, the arguments that used this claim could be reworded to just use the fact that an element that is not irreducible is a product of two nonunits, and follow the same strategy. Alternatively, you could use the hypothesis to show first that every nonunit is divisible by an irreducible.
- Note that the first statement of #2 also applies for nonNoetherian rings, so we don't want to assume Noetherian for it. For example, every element in $K[x_1, x_2, x_3, \dots]$ admits a factorization into irreducibles. Think about this ring has ACC for principal ideals.
- Let $A \subseteq R$ be rings. The condition that a map $\pi : R \to A$ is A-linear is very different from the condition that $\pi : R \to A$ is a ring homomorphism. In checking problem #3 and #4a, these got mixed up sometimes.

  A-linear says $\pi(ar) = a\pi(r)$ for $a \in A, r \in R$, and $\pi(\sum a_i r_i) = \sum a_i \pi(r_i)$ for $a_i \in A, r_i \in R$ (and additive).

  Ring homomorphism says $\pi(ar) = \pi(a)\pi(r)$ for $a \in A, r \in R$, and $\pi(\sum a_i r_i) = \sum \pi(a_i)\pi(r_i)$ for $a_i \in A, r_i \in R$ (and additive).

  Here is an example to distinguish these: let $A = K[x^2] \subseteq R = K[x]$, and $\pi : R \to A$ be the map that sends a polynomial to the sum of its even degree pieces. This is A-linear: if $f \in A$ and $r \in R$, then $\pi(ar)$ is the sum of even degree parts of ar, which is a times the sum of even degree parts of r, which is $a\pi(r)$. However, this is not a ring homomorphism: the even degree parts of a product of two polynomials is not the product of their even degree parts (try x and x).
- To be a *direct summand* means $A \subseteq R$ and there is some $\pi : R \to A$ that is A-linear such that $\pi|_A = 1_A$. It is important to check that a supposed splitting is A-linear. Let's see what happens if we mess with this condition:

  If $A \subseteq R$, then there is always a function $\pi : R \to A$ such that $\pi|_A = 1_A$, so just finding a function means nothing.

  If $A \subseteq R$, it is much more restrictive to ask for an ring map $\pi : R \to A$ such that $\pi|_A = 1_A$. This cannot happen for $A = K[x^2] \subseteq R = K[x]$, for example; we would have $x^2 \mapsto x^2$, so a homomorphism would require $x \mapsto$ some element whose square is x^2, but no such element lives in A.

  Here is a nonexample of being a direct summand: $A = K[x^2, x^3] \subseteq R = K[x]$. If A were a direct summand of R, we would have $R = A \oplus C$ as A-modules. Since this decomposition respects the A-module structure, it respects the K-vector space structure by restriction of scalars as well, because $K \subseteq A$. Thus, $C = K \cdot x$ is a one-dimensional vector space. But, C is a submodule of R, so it is a torsion free A-module; this is a contradiction.
- Also, when we assert that $A \subseteq R$ is a direct summand, we consider the inclusion map to be fixed.
- For #4b, some of us gave basically the same proof for finite generation using Artin-Tate as in class. What I had in mind was the following: R^G is Noetherian (by #4a), and $\mathbb{N}$-graded with $[R^G]_0 = K$, so R^G is finitely generated as an algebra over K by a theorem from class on graded rings.

  The point of having this other argument is that we can apply it to other rings of invariants: if G acts by degree-preserving K-algebra maps, and R^G is a direct summand (which can be true for some infinite groups), then R^G is finitely generated as a K-algebra.