Math 614, Fall 2018, Homework #1

Please write up and turn in at least *five* of the following problems at the beginning of class Thursday, September 20. You are strongly encouraged to work out the rest of them, as well.

(1) (a) Let $R = \mathbb{C}[x]$ and $S = \mathbb{C}[x,y]/(xy)$. Check that $R \subseteq S$, and find a generating set for S as an R-module.
(b) With S as above, find an element $f \in S$ such that S is a finitely generated $A = \mathbb{C}[f]$-module. Can you give an explicit description of the A-module structure of S?

(2) Show that, in a domain R, if every ascending chain of *principal ideals* stabilizes, then every element factors as a product of irreducibles. Hence, in a Noetherian ring, such factorizations exist.

(3) Let $i : R \to S$ be a ring homomorphism. We say that R is a *direct summand* of S if there is an R-module homomorphism $\pi : S \to R$ such that $\pi \circ i$ is the identity on R. Note that this implies that i is injective; we identify $i(R)$ with $R \subseteq S$. Let R be a direct summand of S.
(a) Show that, if I is an ideal of R, that $IS \cap R = I$, where $IS = \sum_j a_j s_j : a_j \in I, s_j \in S$ is the ideal generated by I in S.
(b) Show that if S is Noetherian, then R is Noetherian as well.

(4) Let G be a finite group and R be a polynomial ring over K.[1] Suppose that G acts on R by degree preserving automorphisms, and that the integer $|G|$ is a unit in K.[2]
(a) Show that R^G is a direct summand of R.
(b) Use this to give another proof in this setting that R^G is finitely generated over K.
It turns out that for many important infinite groups G in characteristic zero (including $G = (K^\times)^n, \mathrm{GL}_n(K), \mathrm{SL}_n(K)$, etc.), R^G is a direct summand of R for every representation of G, and hence R^G is finitely generated over K. This is the general idea of Hilbert's proof of finite generation of invariants over $\mathrm{SL}_n(K)$.

(5) The *ring of complex analytic germs* in d variables, denoted $\mathbb{C}\{z_1, \dots, z_d\}$, is the subring of $\mathbb{C}[\![z_1, \dots, z_d]\!]$ consisting of power series that converge on some ball containing the origin.
- A *Weierstrass polynomial* of degree t in z_d is a function of the form $z_d^t + f_{t-1} z_d^{t-1} + \cdots + f_0$ with $f_0, \dots, f_{t-1} \in \mathbb{C}\{z_1, \dots, z_{d-1}\}$.
- The *Weierstrass preparation theorem* says that: If $f \in \mathbb{C}\{z_1, \dots, z_d\}$ satisfies $f(0, \dots, 0) = 0$, and $f(0, \dots, 0, z_d) \not\equiv 0$, then there is some *unit* $g \in \mathbb{C}\{z_1, \dots, z_d\}$ and Weierstrass polynomial h in z_d such that $f = gh$.
- The *Weierstrass division theorem* says that if h is a Weierstrass polynomial of degree t in z_d, and $f \in \mathbb{C}\{z_1, \dots, z_d\}$, then $f = ph + q$ for some $p \in \mathbb{C}\{z_1, \dots, z_d\}$, and $q \in \mathbb{C}\{z_1, \dots, z_{d-1}\}[z_d]$ of degree less than t in z_d.

Use the Weierstrass preparation theorem and Weierstrass division theorem to show that $\mathbb{C}\{z_1, \dots, z_d\}$ is Noetherian.[3][4]

[1] Mentioned in class: K is a field and R has finitely many variables here.

[2] This happens whenever K has characteristic zero, for example.

[3] Hint: Induce on d. Show that any nonzero f satisfies the hypothesis of WPT after a linear change of coordinates, and show that any $\mathbb{C}\{z_1, \dots, z_d\}/(f)$ is Noetherian.

[4] A similar proof holds for polynomial and power series rings over fields. We will soon encounter the polynomial analogue of WPT.

(6) (a) Show that if R is a domain, and G is a finite group acting on R, then G acts on the fraction field $\operatorname{frac}(R)$ of R, and $\operatorname{frac}(R^G) = \operatorname{frac}(R)^G$.
(b) We never concluded that we found all of the invariants in our group of order 8 example. Show that we at least found them on the level of fraction fields; i.e., $\operatorname{frac}(R^G) = \mathbb{C}(x^2y^2, x^4 + y^4, xy(x^4 - y^4))$.

(7) Commutative algebra and invariant theory have been very useful in the study of nonnegative solutions to integer linear equations (*linear diophantine equations*). Here is a toy application to indicate this. Let $A \in \operatorname{Mat}_{n\times m}(\mathbb{Z})$, and K be a field.
(a) Show that the ring $R = \bigoplus_{\alpha \in \mathbb{N}^n \cap \ker(A^T)} K \cdot \underline{x}^\alpha$ is a direct summand of the polynomial ring $S = K[x_1, \dots, x_n]$.
(b) Show that, if K is infinite, that $R = S^G$, where $G = (K^\times)^m$ acts on S by $\underline{\lambda} \cdot x_i = \lambda_1^{a_{i1}} \cdots \lambda_m^{a_{im}} x_i$.
(c) Use part (a) to show that $\ker(A^T) \cap \mathbb{N}^n$ is a finitely generated semigroup.

(8) Show that if $d \in \mathbb{N}_{>1}$, then the ring

$$\{r \in \mathbb{Q}(\sqrt{d}) \mid r \text{ is integral over } \mathbb{Z}\} = \begin{cases} \mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4 \\ \mathbb{Z} + \mathbb{Z}\sqrt{d} & \text{if } d \not\equiv 1 \mod 4. \end{cases}$$

(9) Let K be a field, and R be a Noetherian $\mathbb{N}$-graded ring with $R_0 = K$. Show that, for some $d \in \mathbb{N}$, the subring $R^{(d)} := \bigoplus_{i\in\mathbb{N}} R_{di} \subseteq R$ is generated by elements of degree d.

(10) This problem gives a proof that the invariant ring of SL_2 acting on $\mathbb{C}[X_{2\times n}] = \mathbb{C}\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$ is generated by the 2×2 minors $\{\Delta_{ij} ;= \det \begin{bmatrix} x_i & x_j \\ y_1 & y_j \end{bmatrix} \mid i < j\}$ of X, if $\mathbb{C}$ has characteristic zero.

Define for $1 \le i, j \le n$ the *polarization operators* $E_{ij} := x_i \frac{\partial}{\partial x_j} + y_i \frac{\partial}{\partial y_j}$.
(a) Show that each E_{ij} takes SL_2-invariants to SL_2-invariants.
(b) Show that each E_{ij} sends the subalgebra $\mathbb{C}[\{\Delta_{ij} \mid 1 \le i < j \le n\}]$ to itself.
(c) Show that $\mathbb{C}[X_{2\times n}]^{\mathrm{SL}_2}$ admits an $\mathbb{N}^n$-grading induced by the grading $|x_i| = |y_i| = \vec{e_i}$ on $\mathbb{C}[X_{2\times n}]$.
(d) Prove Cappelli's identity:

$$\begin{Vmatrix} E_{jj} + 1 & E_{ij} \\ E_{ji} & E_{ii} \end{Vmatrix} = \begin{Vmatrix} x_i & x_j \\ y_i & y_j \end{Vmatrix} \circ \begin{Vmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{Vmatrix},$$

as differential operators on $K[X_{2\times n}]$, where $\| \star \|$ denotes determinant.[5]
(e) Prove that $\mathbb{C}[X_{2\times n}]^{\mathrm{SL}_2} = \mathbb{C}[\{\Delta_{ij} \mid 1 \le i < j \le n\}]$.

[5]Here, $\begin{Vmatrix} x_i & x_j \\ y_i & y_j \end{Vmatrix}$ is to be interpreted as the operator "multiplication by $\begin{Vmatrix} x_i & x_j \\ y_i & y_j \end{Vmatrix}$."