Math 614, Fall 2018, Homework #5

Please write up and turn in at least *four* of the following problems at the beginning of class Thursday, November 29. You are strongly encouraged to work out the rest of them, as well.

(1) Let R be a Noetherian ring, and I be an ideal. Consider a collection of minimal primary decompositions of I:

$$I = \mathfrak{q}_{1,\alpha} \cap \cdots \cap \mathfrak{q}_{s,\alpha}, \quad \alpha \in \Lambda$$

where, for each α, $\sqrt{\mathfrak{q}_{i,\alpha}} = \mathfrak{p}_i$.

(a) Suppose that $\mathfrak{p}_j$ is not contained in any other associated prime of I, and let $W = R \smallsetminus \bigcup_{i \neq j} \mathfrak{p}_i$. Find some minimal primary decompositions of $I(W^{-1}R) \cap R$.

(b) Show (by induction on s) that if we we take components $\mathfrak{q}_{1,\alpha_1}, \dots, \mathfrak{q}_{s,\alpha_s}$ from different primary decompositions of I, that we can put them together to get a primary decomposition of I; namely $I = \mathfrak{q}_{1,\alpha_1} \cap \cdots \cap \mathfrak{q}_{s,\alpha_s}$.

(2) Let K be a field, and R be a finitely generated graded K-algebra, with $R_0 = K$.

(a) Show that if M is a graded R-module such that $[M]_{<0} = 0$, then $M = (R_+)M$ only if $M = 0$.

(b) Show that if M is a graded R-module such that $[M]_{<0} = 0$, and $m_1, \dots, m_t \in M$ are homogeneous elements such that $M/(R_+)M$ is generated by $\overline{m_1}, \dots, \overline{m_t}$ as a $R/(R_+)$-vector space, then M is generated by $m_1, \dots, m_t$.[1]

(c) Show that homogeneous elements $x_1, \dots, x_n \in R$ form a homogenous system of parameters for R if and only if $K[x_1, \dots, x_n] \subseteq R$ is a homogeneous Noether normalization for R.[2]

(3) Show that if R is a normal domain of characteristic zero that contains a field, and $R \subseteq S$ is module-finite, then R is a direct summand of S.

(4) Show that if R is a domain of characteristic zero that contains a field, then R is a direct summand of every module-finite extension S only if R is normal.

(5) Recall from class that if $\phi : R \subseteq S$ is an inclusion of domains that is algebra-finite, then $\operatorname{im}(\phi^*)$ contains a nonempty open subset of $\operatorname{Spec}(R)$. Show that the hypothesis that the map is algebra-finite is necessary.

(6) Consider a system of polynomial equations and inequations in $\mathbb{C}[x_1, \dots, x_n]$:

$$(\clubsuit) \qquad f_1(\underline{x}) = \cdots = f_a(\underline{x}) = 0, \quad g_1(\underline{x}) \neq 0 \ , \ \dots \ , \ g_b(\underline{x}) \neq 0.$$

Let $A = \mathbb{Z}[\text{all coefficients of } f_1, \dots, f_a, g_1, \dots, g_b] \subseteq \mathbb{C}$; tautologically, we can interpret $(\clubsuit)$ as a system of polynomial equations and inequations with coefficients in A.

(a) Show that $(\clubsuit)$ has a solution over $\mathbb{C}$ if and only if there is a nonempty open subset $U \in \operatorname{Spec}(A)$ such that $(\clubsuit)$ has a solution in the finite field $A/\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \in U$.

[1] Note that we are not assuming that M is finitely generated, so this is a version of NAK that allows us to actually prove that M is finitely generated!

[2] Hint: Apply the previous statement to the graded ring $K[x_1, \dots, x_n]$.

(b) Show that (♣) has no solution over $\mathbb{C}$ if and only if there is a nonempty open subset $U \in \operatorname{Spec}(A)$ such that (♣) has no solution in the finite field $A/\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \in U$.

(7) In the same setting as #4, suppose that all of the coefficients of the f's and g's are algebraic over $\mathbb{Q}$. In this case, show that the following are equivalent:
- (♣) has a solution over $\mathbb{C}$
- (♣) has a solution over $\overline{\mathbb{F}_p}$ for all but finitely many p
- (♣) has a solution over $\overline{\mathbb{F}_p}$ for infinitely many p.

(8) Problems #3 and #6 from Normalization worksheet