Name: Solutions

Problem 1 (4 points)**.** Circle all the true statements, no justification necessary.

(a) $S_3 \cong D_3$.

(b) S_4 has 8 elements. It has $4! = 24$

(c) S_5 is an abelian group. No!

(d) $(1\,2\,3)$ is an odd permutation. even! $(1\,2\,3) = (1\,3)(1\,2)$

(e) $\mathbb{Z}_7^\times$ is a cyclic group. $\mathbb{Z}_7$ is a finite field!

(f) If G and H are groups of order 4, $G \cong H$. $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$

Problem 2 (3 points)**.** Let $f : G \longrightarrow H$ be an injective group homomorphism. Show that for every $g \in G$, $|f(g)| = |g|$.

For all $n \in \mathbb{Z}$, $f(g^n) = f(g)^n$

Moreover, $f(g)^n = e \Leftrightarrow f(g^n) = e \overset{f \text{ injective}}{\Longleftrightarrow} g^n = e$

$|g| =$ smallest positive integer n st $g^n = e$ (∞ if $g^n \neq e$ for all $n \geq 1$)

$=$ smallest positive integer n st $f(g)^n = e$ (∞ if $f(g)^n \neq e$ for all $n \geq 1$

$= |f(g)|$

Problem 3 (3 points)**.** True or false: the function $\mathbb{Z}_9 \overset{f}{\longrightarrow} \mathbb{Z}_9$ given by $x \mapsto 2x$ is a group isomorphism.

True. Proof

1) f is a group homomorphism: for all $x, y \in \mathbb{Z}_9$,

$f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

2) $g : \mathbb{Z}_9 \to \mathbb{Z}_9$ given by $g(x) = 5x$ is the inverse of f:

$gf(x) = g(2x) = 10x = x$

$fg(x) = f(5x) = 10x = x$ for all $x \in \mathbb{Z}_9$