Name:

Problem 0 (3 points). Circle all the pairs $S \subseteq R$ where S is a subring of R.

(a) $\mathbb{Z} \subseteq \mathbb{Q}$. (b) $\mathbb{N} \subseteq \mathbb{Q}$. (c) $\varnothing \subseteq \mathbb{Z}$. (d) $\mathbb{R} \subseteq \mathbb{C}$.

Problem 1 (4 points). Give an example of a *commutative* subring of the ring $M_2(\mathbb{R})$ of 2×2 matrices with real entries.

One possibility: $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{R} \right\}$.

S is closed under addition: $\begin{bmatrix} a & \\ & a \end{bmatrix} + \begin{bmatrix} b & \\ & b \end{bmatrix} = \begin{bmatrix} a+b & \\ & a+b \end{bmatrix} \in S$

S is closed under additive inverses: $-\begin{bmatrix} a & \\ & a \end{bmatrix} = \begin{bmatrix} -a & \\ & -a \end{bmatrix} \in S$

S is closed under multiplication: $\begin{bmatrix} a & \\ & a \end{bmatrix}\begin{bmatrix} b & \\ & b \end{bmatrix} = \begin{bmatrix} ab & \\ & ab \end{bmatrix} \in S$

$1 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \in S$.

S is commutative: $\begin{bmatrix} a & \\ & a \end{bmatrix}\begin{bmatrix} b & \\ & b \end{bmatrix} = \begin{bmatrix} b & \\ & b \end{bmatrix}\begin{bmatrix} a & \\ & a \end{bmatrix}$.

Problem 2 (3 points). True or false? Justify your answer with a proof if it is true or a counterexample if it is false.

Let R be a ring and $a, b \in R$. If $ab = a$, then $ba = a$.

False, e.g.,
In $M_2(\mathbb{R})$, take

$a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $ab = a$, but $ba = b \neq a$.