Math 412. Operation!

DEFINITION: An **operation** on a set S is a function from $S \times S$ to S.

For example, addition and subtraction are operations on set the of integers (or on the set of real numbers). We might write $\star$ for an operation, and write $x \star y$ to indicate the result of applying an operation to (x, y), just as we would with the symbols $+$, $-$, etc.

A **ring** is a set with two operations, which we usually call addition and multiplication, that behave in similar ways to addition and multiplication of numbers. To make this precise, we specify some special abstract properties of operations.

- *Commutativity.* An operation $\star$ is **commutative** if $x \star y = y \star x$ for any $x, y \in S$.
  - Find an example of an operation on the set of 2×2 matrices that *is* commutative, and an example of an operation on the same set that *is not* commutative. Can you think of more than one of each?

> Commutative: Addition of matrices, elementwise multiplication. Noncommutative: Multiplication of matrices, subtraction of matrices.

- *Associativity.* By definition, operations only take two inputs. If we wanted to operate on three things, we would have to choose two to pair first, then throw in the third. An operation $\star$ is **associative** if we get the same result with either grouping: $(x \star y) \star z = x \star (y \star z)$ for any x, y, z in S.
  - Find an example of an operation on $\mathbb{Z}$ that *is* associative, and an example of an operation on $\mathbb{Z}$ that *is not* associative.

> Associative: addition. Nonassociative: subtraction $(3 - (2 - 1) = 2 \neq 0 = 3 - (2 - 1))$.

  - Let S be the set of functions $X \to X$ for some other set X. Prove that the operation on S "composition of functions" is associative.

> Need to check that $(f \circ g) \circ h = f \circ (g \circ h)$. To do this, show these two functions agree when evaluated at any $x \in X$. Both sides equal $f(g(h(x)))$ when evaluated at x, so they are equal as functions.

  - Can you find an example of an operation on a set that *is* associative but *is not* commutative? What about the other way around? [1]

> Associative not commutative: composition of functions (If $f(x) = x^2$ and $g(x) = -x$, then $(f \circ g)(x) = x^2 \neq -x^2 = (g \circ f)(x)$). Commutative not associative: averaging two numbers $(\text{avg}(\text{avg}(2, 4), 6) = 4.5 \neq 3.5 = \text{avg}(2, \text{avg}(4, 6))$).

- *Identity.* An element $e \in S$ is an **identity** for $\star$ if $e \star x = x \star e = x$ for all $x \in S$.
  - Which of the following operations have an identity? If so, what is it:

[1] Hint: Maybe something on the list of operations under "identity" works.

* $*$ addition on the set $\mathbb{R}[x]$ of real polynomials
* $*$ subtraction on the set $\mathbb{R}[x]$ of real polynomials
* $*$ multiplication of 2×2 matrices
* $*$ division of positive real numbers
* $*$ composition of functions
* $*$ averaging two rational numbers
* $*$ maximum of two rational numbers

* $*$ yes, the zero constant polynomial
* $*$ no
* $*$ yes, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
* $*$ no
* $*$ yes, the function $f(x) = x$
* $*$ no
* $*$ no, but if we considered *positive* rational numbers, 0 is an identity

– Prove that any operation has at most one identity.

If e and e' are identities for $\star$, then $e = e \star e' = e'$, where the first equality follows from the fact that e' is the identity, and the second follows from the fact that e is the identity.

- *Inverses.* If $\star$ is an operation with an identity e, then an **inverse** for an element x is another element y such that $x \star y = y \star x = e$.
  – For each of the operations above that has an identity, does it have an inverse? How do you find inverses for your operation?

For addition of polynomials, inverses exist; they are negatives. For multiplication of matrices, inverses do not always exist. For composition of functions, inverses again do not always exist.

We can also describe operations by tables, like we do with $+$ tables and $\times$ tables. Here are some operation tables for operations on the set $\{a, b, c, d\}$. Decide for each whether the operation is commutative, has an identity, and/or has inverses.

♣	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

♢	a	b	c	d
a	a	d	c	b
b	b	a	d	c
c	c	b	a	d
d	d	c	b	a

$\heartsuit$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

$\spadesuit$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Bonus: Can you find natural operations on $\mathbb{Z}_4$ that correspond to these tables?

"max", $-$, $\times$, $+$

DEFINITION: A *ring* is a set R with two operations, denoted "$+$" and "$\times$" such that

- $+$ and $\times$ are both associative,
- $+$ is commutative,
- $+$ has a identity, which we denote 0,
- Every element of R has an inverse for the operation $+$,
- $\times$ has an identity[2], which we denote 1,
- The two operations are related by the *distributive properties*: $a \times (b+c) = a \times b + a \times c$ and $(a+b) \times c = a \times c + b \times c$.

[2]WARNING: Our definition of ring differs from that of the text! Whenever we say ring, we mean a ring with identity in the notation of the book.