

## Math 412. Adventure sheet on Normal Subgroups

**DEFINITION:** A subgroup  $N$  of a group  $G$  is **normal** if for all  $g \in G$ , the left and right  $N$ -cosets  $gN$  and  $Ng$  are the *same* subsets of  $G$ .

**PROPOSITION:** For any subgroup  $H$  of a group  $G$ , we have  $|H| = |gH| = |Hg|$  for all  $g \in G$ .

**THEOREM 8.11:** A subgroup  $N$  of a group  $G$  is **normal** if and only if for all  $g \in G$ ,

$$gNg^{-1} \subseteq N.$$

Here, the set  $gNg^{-1} := \{gn g^{-1} \mid n \in N\}$ .

**NOTATION:** If  $H \subseteq G$  is *any subgroup*, then  $G/H$  denotes the set of left cosets of  $H$  in  $G$ . Its elements are *sets* denoted  $gH$  where  $g \in G$ . Recall that the cardinality of  $G/H$  is called the **index** of  $H$  in  $G$ . We sometimes write  $H \trianglelefteq G$  to indicate that  $H$  is a normal subgroup of  $G$ .

### A. WARMUP

- (1) Let  $2\mathbb{Z}$  be the subgroup of even integers in  $\mathbb{Z}$ . Fix any  $n \in \mathbb{Z}$ . Describe the left coset  $n + 2\mathbb{Z}$  (your answer will depend on the parity of  $n$ ). Describe the right coset  $2\mathbb{Z} + n$ . Is  $2\mathbb{Z}$  a **normal** subgroup of  $\mathbb{Z}$ ? What is its index? Describe the partition of  $\mathbb{Z}$  into left (respectively, right)  $2\mathbb{Z}$ -cosets.
- (2) Let  $K = \langle (23) \rangle \subset \mathcal{S}_3$ . Find the right coset  $K(1\ 2)$ . Find the left coset  $(1\ 2)K$ . Is  $K$  a normal subgroup of  $\mathcal{S}_3$ ?
- (3) Let  $N = \langle (123) \rangle \subset \mathcal{S}_3$ . Find the right coset  $N(1\ 2)$ . Find the left coset  $(1\ 2)N$ . Describe the partition of  $\mathcal{S}_3$  into left  $N$ -cosets. Compare to the partition into right  $N$ -cosets. Is  $gN = Ng$  for all  $g \in \mathcal{S}_3$ ? Is  $N$  a normal subgroup of  $\mathcal{S}_3$ ?

### Solution.

- (1) The left coset  $n + 2\mathbb{Z}$  is the set of odd numbers if  $n$  is odd and the set of even numbers if  $n$  is even. Ditto for the right coset  $2\mathbb{Z} + n$ . The subgroup  $2\mathbb{Z}$  is normal because  $n + 2\mathbb{Z} = 2\mathbb{Z} + n$  for all  $n$ . Index is two.
- (2) Right coset  $K(1\ 2)$  is  $\{(12), (132)\}$ . Left coset  $(1\ 2)K$  is  $\{(12), (123)\}$ . Since  $K(1\ 2) \neq (1\ 2)K$ ,  $K$  is not normal.
- (3) The right coset  $N(1\ 2) = \{(12), (23), (13)\}$ . The left coset is  $(12)N = \{(12), (23), (13)\}$ . We see that  $(12)N = N(12)$ . To find the partition into left cosets, we compute all left cosets. The only other coset is  $eN = \{e, (123), (132)\}$ . We know this because all left cosets have the same cardinality, so since the coset  $(12)N$  has three elements, so do all others. But the cosets are disjoint! So there can be only one more coset, and it is  $eN$ . The partition into right cosets is the same! We know that  $eN = Ne$ , so one left/right coset is the set  $\{e, (123), (132)\}$ . The other left/right coset is the complement:  $N(12) = (12)N = \{(12), (23), (13)\}$ , which we can also write  $N(13) = (13)N = N(23) = (23)N$ . So yes,  $gN = Ng$  for all  $g$ . So  $N$  is normal.

### B. EASY PROOFS

- (1) Prove that if  $G$  is abelian, then *every subgroup*  $K$  is normal.
- (2) Prove that for any subgroup  $K$ , and any  $g \in K$ , we have  $gK = Kg$ .
- (3) Find an example of subgroup  $H$  of  $G$  which is normal but *does not satisfy*  $hg = gh$  for all  $h \in H$  and all  $g \in G$ .

**Solution.**

- (1) Take arbitrary  $g \in G$ . If  $G$  is abelian, we know  $gK = \{gk \mid k \in K\} = \{kg \mid k \in K\} = Kg$ . So  $K$  is normal.
- (2) If  $g \in K$ , then  $gk \in K$  for all  $k \in K$ , so  $gK \subseteq K$ . But also every  $k \in K$  can be written  $k = g(g^{-1}k) \in gK$ , since  $g^{-1}k \in K$  implies  $g^{-1}k \in K$ . So  $K = gK$ . A similar argument shows that  $K = Kg$ , so  $Kg = gK$  for all  $g \in K$ .
- (3) We saw an example already in A3.

C. Let  $G$  be the group  $(S_5, \circ)$ . Use Theorem 8.11 to determine which of the following are **normal** subgroups.

- (1) The trivial subgroup  $e$ .
- (2) The whole group  $S_5$ .
- (3) The subgroup  $A_5$  of *even* permutations.
- (4) The subgroup  $H$  generated by  $(1\ 2\ 3)$ .
- (5) The subgroup  $S_4$  of permutations that fix 5.
- (6) Use Lagrange's Theorem to compute the index of each subgroup in (1)–(5).

**Solution.**

- (1) The trivial subgroup is normal.
- (2) The whole group is a normal subgroup.
- (3) The group  $A_n$  is normal, because given any  $g \in S_n$ , and any  $h \in A_n$ , we need to check that  $ghg^{-1} \in A_n$ . But  $h$  is a composition of an even number of transpositions, say  $2k$  transpositions, and if  $g$  is a composition of  $d$  transpositions, then so is  $g^{-1}$ . So  $ghg^{-1}$  is a composition of  $d + 2k + d = 2(d + k)$  transpositions, and hence is in  $A_n$ .
- (4) The group  $H = \{e, (123), (132)\}$  is not normal if  $n \geq 4$ : if we conjugate by  $(14)$ , we get  $(14)(123)(14) = (423)$  which is not in  $H$ .
- (5) The subgroup  $S_{n-1}$  of permutations that fix  $n$  is not normal: the element  $(12)$  is in  $S_{n-1}$  but its conjugate by  $(1n)$  is  $(2n)$  which does not fix  $n$ .
- (6) Lagrange's theorem tells us that  $[S_n : A_n] = 2$  (we have even and odd permutations for the cosets)  $[S_n : H] = n!/3$ , and  $[S_n : S_{n-1}] = n!/(n-1)! = n$ .

D. Let  $G \xrightarrow{\phi} H$  be a group homomorphism.

- (1) Prove that the kernel of  $\phi$  is a *normal subgroup* of  $G$ .
- (2) Prove that the group  $SL_n(\mathbb{Q})$  of determinant one matrices with entries in  $\mathbb{Q}$  is a normal subgroup of  $GL_n(\mathbb{Q})$ .

**Solution.** Take  $k \in \ker(\phi)$  and  $g \in G$  arbitrary. We need to show that  $gkg^{-1} \in \ker(\phi)$ . Apply  $\phi$  to get  $\phi(g)\phi(k)\phi(g^{-1})$ . Since  $k \in \ker(\phi)$ , this is  $\phi(g)e\phi(g^{-1}) = \phi(gg^{-1}) = \phi(e_G) = e_H$ . So  $gkg^{-1} \in \ker(\phi)$ .

For the second part, we only need to note that this is the kernel of the group homomorphism  $\det$ .

E. CONJUGATION. Let  $G$  be a group, and  $g, h \in G$ . We call the element  $ghg^{-1}$  is the **conjugate** of  $h$  by  $g$ . Let  $c_g : G \rightarrow G$  be the function given by the rule  $c_g(h) = ghg^{-1}$ . We call this function **conjugation by  $g$** .

- (1) Show that, if  $h_1, h_2 \in G$ , then  $c_g(h_1)c_g(h_2) = c_g(h_1h_2)$ . Thus,  $c_g$  is a *group homomorphism* from  $G$  to itself.
- (2) Show that  $c_{g^{-1}} \circ c_g = c_g \circ c_{g^{-1}}$  is the identity on  $G$ . Conclude that  $c_g$  is an **automorphism** of  $G$ : a group isomorphism from  $G$  to itself.
- (3) Let  $G = \mathcal{S}_n$ , and  $h = (ab)$  be a 2-cycle. What is  $c_g(h)$ ?<sup>1</sup> If instead  $h = (a_1 a_2 \cdots a_t)$  is a  $t$ -cycle, what do you think  $c_g(h)$  is? If you know how to write  $h$  as a product of disjoint cycles, how can you write  $c_g(h)$  as a product of disjoint cycles?
- (4) Interpret the last problem as follows:  $c_g(h)$  is “the same permutation as  $h$  up to relabeling the elements  $\{1, \dots, n\}$  by  $g$ .”
- (5) Now let  $G = \text{GL}_n(\mathbb{R})$ . If  $g = S$  and  $h = A$  are matrices in  $G$ , explain what is the geometric meaning of  $c_g(h)$ . Compare with the previous part.

**Solution.**

(1)

$$c_g(h_1)c_g(h_2) = (gh_1g^{-1})(gh_2g^{-1}) = gh_1(g^{-1}g)h_2g^{-1} = gh_1eh_2g^{-1} = gh_1h_2g^{-1} = c_g(h_1h_2)$$

(2)

$$(c_{g^{-1}} \circ c_g)(h) = c_{g^{-1}}(c_g(h)) = c_{g^{-1}}(ghg^{-1}) = g^{-1}(ghg^{-1})g = (g^{-1}g)h(g^{-1}g) = h$$

so  $c_{g^{-1}} \circ c_g$  is the identity map. Applying the same argument with  $g^{-1}$  in place of  $g$  shows the other composition is the identity.

(3) In the homework, we proved that  $g(ab)g^{-1} = (g(a)g(b))$ . This generalizes to  $g(a_1 a_2 \cdots a_t)g^{-1} = (g(a_1)g(a_2) \cdots g(a_t))$ . To check the equality, we deal with different cases. If  $i = g(a_j)$  for some  $j < t$ , then

$$(g(a_1 a_2 \cdots a_t)g^{-1})(i) = (g(a_1 a_2 \cdots a_t)g^{-1})g(a_j) = (g(a_1 a_2 \cdots a_t))(a_j) = g(a_{j+1}).$$

If  $i = g(a_t)$ , then

$$(g(a_1 a_2 \cdots a_t)g^{-1})(i) = (g(a_1 a_2 \cdots a_t)g^{-1})g(a_t) = (g(a_1 a_2 \cdots a_t))(a_t) = g(a_1).$$

If  $i \neq g(a_j)$  for any  $j = 1, \dots, t$ , then  $g^{-1}(i) \neq a_j$  for any  $j = 1, \dots, t$ , so

$$(g(a_1 a_2 \cdots a_t)g^{-1})(i) = (g(a_1 a_2 \cdots a_t)g^{-1})(g^{-1}(i)) = g(g^{-1}(i)) = i.$$

Thus, this permutation agrees with the  $t$ -cycle  $(g(a_1)g(a_2) \cdots g(a_t))$ .

Finally, since conjugation respects products, given a product of cycles, we can use the last rule to write the conjugate as a product of cycles (of the same lengths).

(4) OK!

(5) We called conjugation *similarity* in linear algebra. It corresponds to change of basis. The matrix  $c_g(h)$  gives the same linear transformation as  $h$  in the basis corresponding to the columns of  $g$ .

F. THE PROOF OF THEOREM 8.11. Let  $G$  be a group and  $H$  some subgroup. Prove that the following are equivalent by showing (1) implies (2) implies (3) implies (4) implies (5) implies (1).

- (1)  $H$  is normal.
- (2)  $gHg^{-1} \subseteq H$  for all  $g \in G$ .
- (3)  $g^{-1}Hg \subseteq H$  for all  $g \in G$ .
- (4)  $g^{-1}Hg = H$  for all  $g \in G$ .
- (5)  $gHg^{-1} = H$  for all  $g \in G$ .

<sup>1</sup>Hint: You did this on the homework, so just remember it instead of re-proving it.

**Solution.** (1)  $\Rightarrow$  (2): If  $H$  is normal, then for every  $h \in H$  and every  $g \in G$ ,  $gh \in Hg$ , so there exists  $h' \in H$  such that  $gh = h'g$ , and  $ghg^{-1} = h' \in H$ .

(2)  $\Rightarrow$  (3): Since the statement holds for any  $g \in G$ , it holds for  $g^{-1}$ .

(2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4): Our proof that (2)  $\Rightarrow$  (3) also gives (3)  $\Rightarrow$  (2), which together make (4).

(4)  $\Leftrightarrow$  (5): Replace  $g$  by  $g^{-1}$ , which we can do since the statements are written over all  $g$ .

(5)  $\Leftrightarrow$  (1): given any  $g \in G$  and any  $h \in H$ ,  $ghg^{-1} = h' \in H$ , so  $gh = h'g^{-1} \in Hg$ ; this shows that  $gH \subseteq Hg$  for all  $g \in G$ . Similarly we can show that  $Hg \subseteq gH$  for all  $g \in G$ , and thus  $H$  is normal.

G. Suppose that  $H$  is an index two subgroup of  $G$ .

- (1) Prove that the partition of  $G$  up into left cosets is the disjoint union of  $H$  and  $G \setminus H$ .
- (2) Prove that the partition of  $G$  up into right cosets is the disjoint union of  $H$  and  $G \setminus H$ .
- (3) Prove that for every  $g \in G$ ,  $gH = Hg$ .
- (4) Prove the THEOREM: *Every subgroup of index two in  $G$  is normal.*

**Solution.**

- (1) If the index is two, there are only two left cosets. Since one is  $H = eH$ , the other is its complement  $G \setminus H$  (as they are disjoint).
- (2) Ditto.
- (3) Consider any  $g \in G$ ; either  $g \in H$  or  $g \notin H$ . If  $g \in H$ , then  $gH = H = Hg$ , since  $H$  is closed under multiplication and  $gH$ . If  $g \notin H$ ,  $gH \neq H$ , so  $gH = G \setminus H$ . Similarly,  $Hg = G \setminus H$ , so  $gH = Hg$ .
- (4) We just saw that  $gH = Hg$  for all  $g \in G$ . By definition,  $H$  is normal.

H. OPERATIONS ON COSETS: Let  $(G, \circ)$  be a group and let  $N \subseteq G$  be a *normal* subgroup.

- (1) Explain why  $Ng = gN$ . Explain why both cosets contain  $g$ .
- (2) Take arbitrary  $ng \in Ng$ . Prove that there exists  $n' \in N$  such that  $ng = gn'$ .
- (3) Take any  $x \in g_1N$  and any  $y \in g_2N$ . Prove that  $xy \in g_1g_2N$ .
- (4) Define a binary operation  $\star$  on the set  $G/N$  of left  $N$ -cosets as follows:

$$G/N \times G/N \rightarrow G/N \quad g_1N \star g_2N = (g_1 \circ g_2)N.$$

Think through the meaning: the elements of  $G/N$  are *sets* and the operation  $\star$  combines two of these sets into a third set: how? Explain why the binary operation  $\star$  is **well-defined**. Where are you using normality of  $N$ ?

- (5) Prove that the operation  $\star$  in (4) is associative.
- (6) Prove that  $N$  is an identity for the operation  $\star$  in (4).
- (7) Prove that every coset  $gN \in G/N$  has an inverse under the operation  $\star$  in (4).
- (8) Conclude that  $(G/N, \star)$  is a group.

**Solution.** This is 8.3 in the book. More next time. Please read 8.3 carefully, and then try this exercise on your own!