Math 412. Adventure sheet on Normal Subgroups

DEFINITION: A subgroup N of a group G is **normal** if for all $g \in G$, the left and right N-cosets gN and Ng are the *same* subsets of G.

PROPOSITION: For any subgroup H of a group G, we have $|H| = |gH| = |Hg|$ for all $g \in G$.

THEOREM 8.11: A subgroup N of a group G is **normal** if and only if for all $g \in G$,

$$gNg^{-1} \subseteq N.$$

Here, the set $gNg^{-1} := \{gng^{-1} \mid n \in N\}$.

NOTATION: If $H \subseteq G$ is *any subgroup*, then G/H denotes the set of left cosets of H in G. It elements are *sets* denoted gH where $g \in G$. Recall that the cardinality of G/H is called the **index** of H in G. We sometimes write $H \trianglelefteq G$ to indicate that H is a normal subgroup of G.

A. WARMUP

(1) Let $2\mathbb{Z}$ be the subgroup of even integers in $\mathbb{Z}$. Fix any $n \in \mathbb{Z}$. Describe the left coset $n + 2\mathbb{Z}$ (your answer will depend on the parity of n). Describe the right coset $2\mathbb{Z} + n$. Is $2\mathbb{Z}$ a **normal** subgroup of $\mathbb{Z}$? What is its index? Describe the partition of $\mathbb{Z}$ into left (respectively, right) $2\mathbb{Z}$-cosets.
(2) Let $K = \langle (2\,3) \rangle \subset \mathcal{S}_3$. Find the right coset $K(1\ 2)$. Find the left coset $(1\ 2)K$. Is K a normal subgroup of $\mathcal{S}_3$?
(3) Let $N = \langle (1\,2\,3) \rangle \subset S_3$. Find the right coset $N(1\ 2)$. Find the left coset $(1\ 2)N$. Describe the partition of $\mathcal{S}_3$ into left N-cosets. Compare to the partition into right N-cosets. Is $gN = Ng$ for *all* $g \in \mathcal{S}_3$? Is N a normal subgroup of $\mathcal{S}_3$?

B. EASY PROOFS

(1) Prove that if G is abelian, then *every subgroup* K is normal.
(2) Prove that for any subgroup K, and any $g \in K$, we have $gK = Kg$.
(3) Find an example of subgroup H of G which is normal but *does not satisfy* $hg = gh$ for all $h \in H$ and all $g \in G$.

C. Let G be the group $(S_5, \circ)$. Use Theorem 8.11 to determine which of the following are **normal** subgroups.

(1) The trivial subgroup e.
(2) The whole group S_5.
(3) The subgroup A_5 of *even* permutations.
(4) The subgroup H generated by $(1\,2\,3)$.
(5) The subgroup S_4 of permutations that fix 5.
(6) Use Lagrange's Theorem to compute the index of each subgroup in (1)–(5).

D. Let $G \xrightarrow{\phi} H$ be a group homomorphism.

(1) Prove that the kernel of ϕ is a *normal subgroup* of G.
(2) Prove that the group $SL_n(\mathbb{Q})$ of determinant one matrices with entries in $\mathbb{Q}$ is a normal subgroup of $GL_n(\mathbb{Q})$.

E. CONJUGATION. Let G be a group, and $g, h \in G$. We call the element ghg^{-1} is the **conjugate** of h by g. Let $c_g : G \to G$ be the function given by the rule $c_g(h) = ghg^{-1}$. We call this function **conjugation by** g.

(1) Show that, if $h_1, h_2 \in G$, then $c_g(h_1)c_g(h_2) = c_g(h_1h_2)$. Thus, c_g is a *group homomorphism* from G to itself.
(2) Show that $c_{g^{-1}} \circ c_g = c_g \circ c_{g^{-1}}$ is the identity on G. Conclude that c_g is an **automorphism** of G: a group isomorphism from G to itself.
(3) Let $G = \mathcal{S}_n$, and $h = (\,a\,b\,)$ be a 2-cycle. What is $c_g(h)$?[1] If instead $h = (\,a_1\,a_2\,\cdots\,a_t\,)$ is a t-cycle, what do you think $c_g(h)$ is? If you know how to write h as a product of disjoint cycles, how can you write $c_g(h)$ as a product of disjoint cycles?
(4) Interpret the last problem as follows: $c_g(h)$ is "the same permutation as h up to relabeling the elements $\{1, \ldots, n\}$ by g."
(5) Now let $G = \mathrm{GL}_n(\mathbb{R})$. If $g = S$ and $h = A$ are matrices in G, explain what is the geometric meaning of $c_g(h)$. Compare with the previous part.

F. THE PROOF OF THEOREM 8.11. Let G be a group and H some subgroup. Prove that the following are equivalent by showing (1) implies (2) implies (3) implies (4) implies (5) implies (1).

(1) H is normal.
(2) $gHg^{-1} \subseteq H$ for *all* $g \in G$.
(3) $g^{-1}Hg \subseteq H$ for all $g \in G$.
(4) $g^{-1}Hg = H$ for all $g \in G$.
(5) $gHg^{-1} = H$ for all $g \in G$.

G. Suppose that H is an index two subgroup of G.

(1) Prove that the partition of G up into left cosets is the disjoint union of H and $G \setminus H$.
(2) Prove that the partition of G up into right cosets is the disjoint union of H and $G \setminus H$.
(3) Prove that for every $g \in G$, $gH = Hg$.
(4) Prove the THEOREM: *Every subgroup of index two in G is normal.*

H. OPERATIONS ON COSETS: Let $(G, \circ)$ be a group and let $N \subseteq G$ be a *normal* subgroup.

(1) Explain why $Ng = gN$. Explain why both cosets contain g.
(2) Take arbitrary $ng \in Ng$. Prove that there exists $n' \in N$ such that $ng = gn'$.
(3) Take any $x \in g_1N$ and any $y \in g_2N$. Prove that $xy \in g_1g_2N$.
(4) Define a binary operation $\star$ on the set G/N of left N-cosets as follows:

$$G/N \times G/N \to G/N \qquad g_1N \star g_2N = (g_1 \circ g_2)N.$$

Think through the meaning: the elements of G/N are *sets* and the operation $\star$ combines two of these sets into a third set: how? Explain why the binary operation $\star$ is **well-defined.** Where are you using normality of N?
(5) Prove that the operation $\star$ in (4) is associative.
(6) Prove that N is an identity for the operation $\star$ in (4).
(7) Prove that every coset $gN \in G/N$ has an inverse under the operation $\star$ in (4).
(8) Conclude that $(G/N, \star)$ is a group.

[1]Hint: You did this on the homework, so just remember it instead of reproving it.