Name: Solutions

Math 412 Winter 2019 Midterm Exam

Time: 100 mins.

1. Answer each question in the space provided. If you require more space, you may use the blank page at the end of this exam, but you must clearly indicate in the provided answer space that you have done so.
2. You may use any results proved in class, on the homework, or in the textbook, except for the specific question being asked. You should clearly state any facts you are using.
3. Remember to show all your work.
4. No calculators, notes, or other outside assistance allowed.

Best of luck!

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Problem 1 (20 points)**.** Write complete, precise definitions for, or precise mathematical characterizations of, each of the following italicized terms. Be sure to include any quantifiers as needed.

a) An *ideal* I in a ring R.

A nonempty subset of R that is closed for sums, and such that for all $r \in I$, $s \in R$, $sr, rs \in I$.

b) A *field* $\mathbb{F}$.

A commutative ring $\mathbb{F} \neq \{0\}$ such that every non-zero element in $\mathbb{F}$ is invertible.

c) The *additive inverse* y of an element x in a ring R.

An element $y \in R$ such that $y + x = x + y = 0$.

d) The *congruence class* of a given integer n modulo 17.

$\{x \in \mathbb{Z} : 17 \mid (x - n)\}$ or $\{n + 17k : k \in \mathbb{Z}\}$

Problem 2 (12 points)**.** For each of the questions below, give an example with the required properties. No explanations required.

a) A domain that is not a field.

$\mathbb{Z}$

b) A surjective ring homomorphism that is not injective.

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}_2 \\ n &\longmapsto [n]_2 \end{aligned}$$

c) A nonzero nonunit (element that is not a unit) in $\mathbb{Z}_{4699}$.

Note: $4699 = 127 \cdot 37$ is a prime factorization.

$[127]_{4699}$.

d) A finite subring of an infinite ring.

$\mathbb{Z}_2 \subseteq \mathbb{Z}_2[x]$

Problem 3 (20 points)**.** For each of the questions below, indicate clearly whether the statement is *true* or *false*, and give a short justification.

a) There are 11 distinct principal ideals in the ring $\mathbb{Z}_{11}$.

False. there are only 2 distinct (principal) ideals in $\mathbb{Z}_{11}$.
(0) and $(1) = (2) = (3) = (4) = (5) = (6) = (7) = (8) = (9) = (10)$

b) Every subring of a commutative ring is commutative.

True. If S is a subring of the commutative ring R,
then for all $a, b \in S$, $ab = ba$ in R
so
$ab = ba$ in S.

c) If $\mathbb{F}$ is a field, then the polynomial ring $\mathbb{F}[x]$ is a field.

False. x is not invertible.

d) If R is a ring in which $0_R \neq 1_R$, and $\varphi : M_2(\mathbb{R}) \to R$ is a homomorphism, then φ must be injective.

True.
$\ker \varphi$ is an ideal in $M_2(\mathbb{R})$
the only ideals in $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$
By definition, $\varphi(Id_2) = 1_R \neq 0_R$.
then $\ker \varphi = \{0\}$ and φ is injective.

Problem 4 (10 points). Consider the following operation table for an associative operation $*$ on the set $S = \{a, b, c, d, e, f\}$: here the entry in row x and column y corresponds to the value of $x * y$. Use the table to answer the following questions.

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	e	f	d
c	c	a	b	f	d	e
d	d	f	e	a	c	b
e	e	d	f	b	a	c
f	f	e	d	c	b	a

a) Does $*$ have an identity, and if so, what is it?

Yes, a, since $a * x = x * a = x$ for all $x \in S$.

b) Is the operation $*$ commutative?

No, since $d * b = f \neq e = b * d$.

c) Can the operation $*$ be the multiplication for some ring? Justify your answer.

No! If $*$ were the multiplication in some ring, there would be a row and a column all with the same element, 0, since then we would have

$$0 * x = x * 0 \quad \text{for all } x \in S.$$

Problem 5 (12 points)**.** For each of the following elements of various rings, find a multiplicative inverse, or else explain why no multiplicative inverse exists.

a) $[26]_{57} \in \mathbb{Z}_{57}$.

$$57 = 2 \cdot 26 + 5$$
$$26 = 5 \cdot 5 + 1 \quad \Longrightarrow \quad \gcd(26, 57) = 1, \text{ and } 1 = 11 \cdot 26 - 5 \cdot 57$$
$$5 = 5 \cdot 1 + 0$$

so $[11]_{57} \cdot [26]_{57} = [1]_{57}$ and $[11]_{57}$ is the inverse of $[26]_{57}$

b) $2x + 1 \in \mathbb{Z}[x]$.

the units in $\mathbb{Z}[x]$ are the units in $\mathbb{Z}$, meaning ± 1.
So $2x+1$ is not invertible.

c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$.

$$\frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

check:
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 6 (10 points). Consider $\mathcal{L} = \left\{ \begin{bmatrix} a & 0 \\ 2b & c \end{bmatrix} : a, b, c \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$.

a) Show that $\mathcal{L}$ is a subring of $M_2(\mathbb{Z})$.

$\begin{bmatrix} 0 & 0 \\ 2\cdot 0 & 0 \end{bmatrix} \in \mathcal{L}$, $\begin{bmatrix} 1 & 0 \\ 2\cdot 0 & 1 \end{bmatrix} \in \mathcal{L}$

$\mathcal{L}$ is closed under sums: $\begin{bmatrix} a & 0 \\ 2b & c \end{bmatrix} + \begin{bmatrix} d & 0 \\ 2e & f \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 2(b+e) & c+f \end{bmatrix} \in \mathcal{L}$

$\mathcal{L}$ is closed under products: $\begin{bmatrix} a & 0 \\ 2b & c \end{bmatrix} \begin{bmatrix} d & 0 \\ 2e & f \end{bmatrix} = \begin{bmatrix} ad & 0 \\ 2(bd+ce) & cf \end{bmatrix} \in \mathcal{L}$

$\mathcal{L}$ is closed under additive inverses: $-\begin{bmatrix} a & 0 \\ 2b & c \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 2(-b) & -c \end{bmatrix} \in \mathcal{L}$

b) Let I be the ideal of $\mathcal{L}$ of matrices with zeroes on the diagonal:

$$I = \left\{ \begin{bmatrix} 0 & 0 \\ 2b & 0 \end{bmatrix} : b \in \mathbb{Z} \right\} \subseteq \mathcal{L}.$$

Consider the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}.$$

Show that $(A+I)^2 = 1 + I$ in $\mathcal{L}/I$.

$A^2 = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$A^2 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix} \in I \implies A^2 + I = 1 + I$

So $(A+I)^2 = A^2 + I = 1 + I$.

Problem 7 (6 points)**.** Let $f(x), h(x), j(x) \in \mathbb{R}[x]$. Prove that[1] if $f(x)g(x) \equiv h(x) \mod j(x)$ has a solution $g(x)$, then $\gcd(f(x), j(x)) \mid h(x)$.

Suppose $f(x)g(x) \equiv h(x) \mod j(x)$ for some $g(x)$. Then

$$f(x)g(x) - h(x) = j(x)\,k(x) \quad \text{for some } g(x), k(x) \in \mathbb{R}[x].$$

then $f(x)g(x) - j(x)k(x) = h(x)$.

Let $d = \gcd(f(x), j(x))$. Since $d \mid f(x)$ and $d \mid j(x)$,

$$d \mid (f(x)g(x) - j(x)k(x)), \quad \text{so} \quad d \mid h(x).$$

[1]Recall that $a(x) \equiv b(x) \mod j(x)$ here means $a(x)$ is congruent to $b(x)$ modulo the ideal $I = (j(x))$ generated by $j(x)$.

Problem 8 (10 points).

a) How many different ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ are there?

None. If $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_4$ is a ring homomorphism, then

$f(1,1) = 1$, but $(1,1)+(1,1) = (0,0)$ and

$0 = f(0,0) = f((1,1)+(1,1)) = f(1,1) + f(1,1) = 1+1 = 2$

But $2 \neq 0$ in $\mathbb{Z}_4$.

b) How many different ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ are there?

If $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_2$ is a ring homomorphism, $f(1,1) = 1$ and $f(0,0) = 0$.

Note $(1,1) = (1,0)+(0,1)$. If $f(1,0) = f(0,1) = 0$, then $f(1,1) = 0+0 \neq 1$ $\lightning$

If $f(0,1) = f(1,0) = 1$, then $f(1,1) = 1+1 = 0 \neq 1$ $\lightning$

So either $f(1,0) = 1$ and $f(0,1) = 0$ or $f(1,0) = 0$ and $f(0,1) = 1$.

We conclude that there are at most 2 ring homomorphisms, which we can rewrite as

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_2 &\xrightarrow{f} \mathbb{Z}_2 \\ (a,b) &\longmapsto a \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_2 &\xrightarrow{g} \mathbb{Z}_2 \\ (a,b) &\longmapsto b \end{aligned}$$

(projection onto the 1st factor) (projection onto the 2nd factor)

these maps are clearly ring homomorphisms!

there are <u>exactly</u> 2 ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$.