Math 412. Comments on Exam 1

1(B). If $1 \neq 0$, then 0 is never a unit, because 0 times anything is 0. In a field, every *nonzero* element is a unit.

3(D). A common mistake in this problem was exhibiting a map that was not injective, but that turned out to not be a ring homomorphism. Be careful to check that if you say something is a homomorphism that it really is! In particular, we have checked before that the determinant map is *not* a ring homomorphism. The map that takes a matrix and returns the entry in a fixed position is also not a ring homomorphism. (Why?)

4(C). To show that this *is* a valid operation table, it is not enough to say that it satisfies the axioms of a ring that pertain directly to multiplication; a ring must have two operations that are compatible via the distributive laws, so the axioms for addition in combination with these may have consequences on what the multiplication can look like. So, to prove that it is a valid operation table, we should *exhibit a ring* that has exactly this multiplication structure. It turns out that this is impossible, since nothing can act like the zero element.

A related question that some people discussed is whether this can be *part of* a multiplication table for a ring. Again, to prove that this is true, we would have to *exhibit a ring* for which the given table described the multiplication on a subset. This *is* possible! You are all encouraged to think about how to construct such a ring; we will return to this later in the semester (if you make sure to remind me to!).

5(C). To say that b is the multiplicative inverse of a means that $ab = 1$ *and* $ba = 1$. It suffices to check only one of these equalities in a commutative ring, since $ab = ba$ for any a, b in such a ring, but it dies not suffice in general. We saw examples in the homework where one holds but not the other!

It turns out that if $\mathbb{F}$ is a field, and $a, b \in M_n(\mathbb{F})$, then $ab = \text{identity}$ if and only if $ba = \text{identity}$. You probably proved this in math 217 for the case $\mathbb{F} = \mathbb{R}$, but we didn't prove anything like this for general fields in this class. See if you can prove it!

6(B). The point of this problem was to check your understanding of quotient rings: what the elements are, and what the operations are. One notable mistake was, essentially, to conflate $(A + I)^2$ with $\{(A + i)^2 \mid i \in I\}$. In general, these are not the same sets! For a more familiar example, consider $[3]_7^2$. This is supposed to be

$$[3]_7[3]_7 = [9]_7 = \{\dots, -12, -5, 2, 9, 16, \dots\},$$

a congruence class in $\mathbb{Z}_7$. Contrast this with

$$\{n^2 \mid n \in [3]_7\} = \{49k^2 + 42k + 9 \mid k \in \mathbb{Z}\} = \{9, 16, 100, 121, \dots\}$$

The latter set is much smaller: it only contains positive integers that are squares. The latter set is a subset of the former (all of these squares are indeed congruent modulo 7), but the sets are not the same. We should correctly use the definitions of elements and operations in the quotient ring here.

7. Knowing that c and d are both linear combinations of a and b does not imply $c|d$ nor $c = d$.

8. The zero map is *never* a ring homomorphism, unless the target ring is $\{0\}$.

8. We proved that there is exactly one ring homomorphism $\mathbb{Z}_n \longrightarrow \mathbb{Z}_m$ if and only if $m|n$, and otherwise there are no ring homomorphisms $\mathbb{Z}_n \longrightarrow \mathbb{Z}_m$. This question was not about ring homomorphisms $\mathbb{Z}_n \longrightarrow \mathbb{Z}_m$, so we cannot use that fact here (at least not directly).

8(A). We know that $0 \mapsto 0$ and $1 \mapsto 1$. A common mistake was saying that this implies that the remaining elements must map to either 2 or 3 — ring homomorphisms are not necessarily surjective!