Math 412 Adventure sheet on permutation groups

DEFINITION: The **symmetric group** $\mathcal{S}_n$ is the group of bijections from any set of n objects, which we usually call simply $\{1, 2, \ldots, n\}$, to itself. An element of this group is called a **permutation** of $\{1, 2, \ldots, n\}$. The group operation in $\mathcal{S}_n$ is *composition* of mappings.

PERMUTATION STACK NOTATION: The notation $\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ denotes the permutation that sends i to k_i for each i.

CYCLE NOTATION: The notation $(a_1\ a_2\ \cdots\ a_t)$ refers to the (special kind of!) permutation that sends a_i to a_{i+1} for $i < t$, a_t to a_1, and fixes any element other than the a_i's. A permutation of this form is called a t**-cycle.** A 2-cycle is also called a **transposition.**

Remember that a cycle is a function, so if we have cycles side-by-side, this refers to composition of functions, where the composition as usual goes from right to left.

THEOREM 7.24: Every permutation can be written as a product of *disjoint cycles* — cycles that all have no elements in common. Disjoint cycles commute.

THEOREM 7.26: Every permutation can be written as a product of *transpositions,* not necessarily disjoint.

A. WARM-UP WITH ELEMENTS OF $\mathcal{S}_n$

(1) Write the permutation $(1\ 3\ 5)(2\ 7) \in \mathcal{S}_7$ in permutation stack notation.
(2) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 2 & 4 & 7 & 5 \end{pmatrix} \in \mathcal{S}_7$ in cycle notation.
(3) If $\sigma = (1\ 2\ 3)(4\ 6)$ and $\tau = (2\ 3\ 4\ 5\ 6)$ in $\mathcal{S}_7$, compute $\sigma\tau$; write your answer in stack notation. Now also write it as a product of disjoint cycles.
(4) With σ and τ as in (4), compute $\tau\sigma$. Is $\mathcal{S}_7$ abelian?
(5) List all elements of $\mathcal{S}_3$ in cycle notation. What is the order of each?
(6) What is the inverse of $(1\ 2\ 3)$? What is the inverse of $(1\ 2\ 3\ 4)$? How about $(1\ 2\ 3\ 4\ 5)^{-1}$?

B. THE SYMMETRIC GROUP $\mathcal{S}_4$

(1) What is the order of $\mathcal{S}_4$?
(2) List all 2-cycles in $\mathcal{S}_4$. How many are there?
(3) List all 3-cycles in $\mathcal{S}_4$. How many are there?
(4) List all 4-cycles in $\mathcal{S}_4$. How many are there?
(5) List all 5-cycles in $\mathcal{S}_4$.
(6) How many permutations in $\mathcal{S}_4$ are not cycles? Find them all.
(7) Find the order of each element in $\mathcal{S}_4$. Why are the orders the same for permutations with the same "cycle type"?
(8) Find cyclic subgroups of $\mathcal{S}_4$ of orders 2, 3, and 4.
(9) Find a subgroup of $\mathcal{S}_4$ isomorphic to the Klein 4-group. List out its elements.
(10) List out all elements in the subgroup $H = \langle (1\ 2\ 3), (2\ 3) \rangle$ of $\mathcal{S}_4$ generated by $(1\ 2\ 3)$ and $(2\ 3)$. What familiar group is this isomorphic to? Can you find four different subgroups of $\mathcal{S}_4$ isomorphic to $\mathcal{S}_3$?

C. EVEN AND ODD PERMUTATIONS. A permutation is **odd** if it is a composition of an odd number of transposition, and **even** if it is a product of an even number of transpositions.

(1) Explain why a definition like this might be problematic. Problem G below justifies this definition.
(2) Write the permutation (123) as a product of transpositions. Is (123) even or odd ?

(3) Write the permutation (1234) as a product of transpositions. Is (1234) even or odd ?
(4) Write the $\sigma = (12)(345)$ a product of transpositions in two different ways. Is σ even or odd ?
(5) Prove that every 3-cycle is an even permutation.

D. THE ALTERNATING GROUPS

(1) Prove that the subset of even permutations in S_n is a subgroup. This is the called the **alternating group** A_n.
(2) List out the elements of A_2. What group is this?
(3) List out the elements of A_3. To what group is this isomorphic?
(4) How many elements in A_4? Is A_4 abelian? What about A_n?

E. THE SYMMETRIC GROUP $\mathcal{S}_5$

(1) Find one example of each type of element in S_5 or explain why there is none:
  (a) A 2-cycle
  (b) A 3-cycle
  (c) A 4-cycle
  (d) A 5-cycle
  (e) A 6-cycle
  (f) A product of disjoint transpositions
  (g) A product of 3-cycle and a disjoint 2-cycle.
  (h) A product of 2 disjoint 3 cycles.
(2) For each example in (1), find the order of the element.
(3) What are all possible orders of elements in $\mathcal{S}_5$?
(4) What are all possible orders of cycle subgroups of $\mathcal{S}_5$.
(5) For each example in (1), write the element as a product of transpositions. Which are even and which are odd?

F. Discuss with your workmates how one might prove Theorem 7.26.[1]

G. PERMUTATION MATRICES. We say that an $n \times n$ matrix is a **permutation matrix** if it has exactly one 1 in each row and each column, and the other entries 0. If $\sigma \in \mathcal{S}_n$ is a permutation, let P_σ be the $n \times n$ permutation matrix with $(\sigma(i), i)$ entry 1 for all i, and all other entries 0.

(1) Show that $P_\sigma e_i = e_{\sigma(i)}$ for any permutation σ, where e_j is the jth standard basis vector.
(2) Show that $P_\sigma P_\tau = P_{\sigma\circ\tau}$.
(3) Show that the set of permutation matrices is a subgroup of $GL_n(\mathbb{R})$ that is isomorphic to $\mathcal{S}_n$.
(4) Show that the determinant of $P_{(ij)}$ is -1.
(5) Show that if σ is a product of an even number of transpositions, then the determimant of P_σ is 1, and if σ is a product of an odd number of transpositions, then the determimant of P_σ is -1. Conclude that the sign of a permutation is well-defined.

[1] Hint: Imagine lining everyone in the class up in a straight line. How can we put the class in alphabetical order by a sequence of swaps?