Math 412. Adventure sheet on Quotient Groups

Fix an arbitrary group $(G, \circ)$.

DEFINITION: A subgroup N of G is **normal** if for all $g \in G$, the left and right N-cosets gN and Ng are the *same* subsets of G.

NOTATION: If $H \subseteq G$ is *any subgroup*, then G/H denotes the set of left cosets of H in G. Its elements are *sets* denoted gH where $g \in G$. The cardinality of G/H is called the **index** of H in G.

DEFINITION/THEOREM 8.13: Let N be a *normal subgroup* of G. Then there is a well-defined binary operation on the set G/N defined as follows:

$$G/N \times G/N \to G/N \qquad g_1N \star g_2N = (g_1 \circ g_2)N$$

making G/N into a group. We call this the **quotient group** "G modulo N".

A. WARMUP: Define the *sign map:*

$$S_n \to \{\pm 1\} \quad \sigma \mapsto 1 \text{ if } \sigma \text{ is even; } \sigma \mapsto -1 \text{ if } \sigma \text{ is odd.}$$

(1) Prove that sign map is a group homomorphism.
(2) Use the sign map to give a different proof that A_n is a normal subgroup of S_n for all n.
(3) Describe the A_n-cosets of S_n. Make a table to describe the quotient group structure S_n/A_n. What is the identity element?

B. OPERATIONS ON COSETS: Let $(G, \circ)$ be a group and let $N \subseteq G$ be a **normal** subgroup.

(1) Take arbitrary $ng \in Ng$. Prove that there exists $n' \in N$ such that $ng = gn'$.
(2) Take any $x \in g_1N$ and any $y \in g_2N$. Prove that $xy \in g_1g_2N$.
(3) Define a binary operation $\star$ on the set G/N of left N-cosets as follows:

$$G/N \times G/N \to G/N \qquad g_1N \star g_2N = (g_1 \circ g_2)N.$$

Think through the meaning: the elements of G/N are *sets* and the operation $\star$ combines two of these sets into a third set: how? Explain why the binary operation $\star$ is **well-defined.** Where are you using normality of N?
(4) Prove that the operation $\star$ in (4) is associative.
(5) Prove that N is an identity for the operation $\star$ in (4).
(6) Prove that every coset $gN \in G/N$ has an inverse under the operation $\star$ in (4).
(7) Conclude that $(G/N, \star)$ is a group.
(8) Does the set of **right cosets** also have a natural group structure? What is it? Does it differ from G/N?

C. EASY EXAMPLES OF QUOTIENT GROUPS:

(1) In $(\mathbb{Z}, +)$, explain why $n\mathbb{Z}$ is a normal subgroup and describe the corresponding quotient group.
(2) For any group G, explain why G is a normal subgroup of itself. What is the quotient G/G?
(3) For any group G, explain why $\{e\}$ is a normal subgroup of G. What is the quotient $G/\{e\}$?

D. ANOTHER EXAMPLE. Let $G = \mathbb{Z}_{25}^{\times}$. Let N be the subgroup generated by $[7]$.

(1) Give a one-line proof that N is normal.
(2) List out the elements of G and of N. Compute the order of both. Compute the index of N in G.
(3) List out the elements of G/N; don't forget that each one is a *coset* (in particular, a set whose elements you should list).
(4) Give each coset in G/N a reasonable name. Now make a multiplication table for the group G/N, using these names. Is G/N abelian?

E. THE CANONICAL QUOTIENT MAP: Prove that the map

$$G \to G/N \qquad g \mapsto gN$$

is a group homomorphism. What is its kernel?

F. INDEX TWO. Suppose that H is an index two subgroup of G. Last time, we proved the

THEOREM: *Every subgroup of index two in G is normal.*

(1) Describe the quotient group G/H. What are its elements? What is the table?
(2) Find an example of an index two subgroup of D_n and describe its two cosets explicitly. Make a table for this group and describe the canonical quotient map $G \to G/H$ explicitly.

G. PRODUCTS AND QUOTIENT GROUPS: Let K and H be arbitrary groups and let $G = K \times H$.

(1) Find a natural homomorphism $G \to H$ whose kernel K' is $K \times e_H$.
(2) Prove that K' is a normal subgroup of G, whose cosets are all of the form $K \times h$ for $h \in H$.
(3) Prove that G/K' is isomorphic to H.

H. What goes wrong if we try to define a group structure on the set of right cosets G/H where H is a *non-normal* subgroup of G? Try illustrating the problem with the non-normal subgroup $\langle (1\,2) \rangle$ in S_3.

I. THE FIRST ISOMORPHISM THEOREM. Conjecture and prove **first isomorphism theorem** for groups.