Homework #8

Problems to hand in on Thursday, March 28, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Let S^1 be the subset of $\mathbb{C}$ consisting of complex numbers of absolute value 1; that is

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\}.$$

(a) Prove that S^1 is a subgroup of $\mathbb{C}^\times$.

(b) Prove that the map

$$S^1 \to \mathrm{SL}_2(\mathbb{R}) \qquad x + iy \mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

is an injective group homomorphism.

(c) Prove that S^1 is isomorphic to $\mathrm{SO}_2(\mathbb{R})$, the group of 2×2 orthogonal matrices of determinant 1. Use this to give a geometric interpretation of the group S^1 that explains why some call it the "continuous rotation group."

(d) For every positive integer n, find an element of order n in S^1.

(e) Find an element of infinite order in S^1.

2) Towards the end of the worksheet on group homomorphisms, we encountered the following:

Theorem: If $\mathbb{F}$ is a finite field, then $\mathbb{F}^\times$ is cyclic.

(a) Check that 2 is not a generator for $\mathbb{Z}_{17}^\times$ but 3 is a generator for $\mathbb{Z}_{17}^\times$.

(b) Verify that $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2 + x + 2)$ is a field, and find a generator for $\mathbb{F}_9^\times$.

(c) Read Corollary 7.10 on page 200, and use this corollary to prove the Theorem above.[1]

(d) The Theorem above only applies to finite fields, but we can sometimes describe multiplicative groups of infinite fields in terms of other groups. Show that $\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}_2$.

(e) Show that $\mathbb{C}^\times \cong \mathbb{R} \times S^1$.

3) Consider the following elements in $GL_2(\mathbb{C})$:

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

Let Q be the subgroup of $GL_2(\mathbb{C})$ generated by the matrices $\mathbf{i}, \mathbf{j}, \mathbf{k}$. You should verify (but not necessarily turn in a proof) that Q contains the 8 elements $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, You may wish to make a multiplication table for Q to answer the following questions. (You do not need to turn in the multiplication table – although notice you have already written in down in last week's webwork!)

(a) Find the complete list of all cyclic subgroups of Q of order 4.

(b) Find the complete list of all cyclic subgroups of Q of order 2.

(c) Find the complete list of all noncyclic subgroups of Q of order 4.

[1] For a hint, look at the worksheet on group homomorphisms.

(d) Can Q be generated by two elements? Prove it.

(e) Is Q_8 isomorphic to D_4? Prove or disprove.

4) Consider the symmetric group $\mathbb{S}_n$.

(a) Show that every element of $\mathbb{S}_n$ is a product of transpositions.[2]

(b) Let $\tau \in \mathbb{S}_n$ be a permutation, and $(\,a\,b\,)$ be a transposition. Show that $\tau(\,a\,b\,)\tau^{-1} = (\,\tau(a)\,\tau(b)\,)$, the transposition changing $\tau(a)$ and $\tau(b)$.

(c) Show that $(\,i\,j\,) = (\,1\,i\,)(\,1\,j\,)(\,1\,i\,)$. Conclude that every element of $\mathcal{S}_n$ is the product of transpositions of the form $(\,1\,i\,)$.

(d) Let σ be the n-cycle $(2\,\cdots\,n\,)$. Show that $(\,1\,i\,) = \sigma^{i-2}(\,1\,2\,)(\sigma^{-1})^{i-2}$. Conclude that $\mathcal{S}_n = \langle(\,1\,2\,),(2\,\cdots\,n\,)\rangle$.

[2] Hint: One possibility for a quick solution is induction on n. Can you multiply any permutation by a transposition to obtain a permutation that fixes one element?