Homework #7

Problems to hand in on Thursday, March 21, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Every group G such that $x^2y^2 = (xy)^2$ for all $x, y \in G$ must be abelian.

2) Every group of order n containing an element of order n is abelian.

3) Let p be a prime integer, and consider the group $\mathrm{GL}_2(\mathbb{Z}_p)$ of invertible 2×2 matrices with entries in $\mathbb{Z}_p$.

   a) Prove that for any nonzero column $\begin{bmatrix} a \\ b \end{bmatrix}$, the matrices $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that are noninvertible are precisely those for which there exists n such that $\begin{bmatrix} c \\ d \end{bmatrix} = n \begin{bmatrix} a \\ b \end{bmatrix}$.

   b) Show that the set of upper triangular invertible matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ forms a subgroup of order $p(p-1)^2$, which is non-abelian when $p \neq 2$.

   c) Compute the order of $\mathrm{GL}_2(\mathbb{Z}_p)$.

   d) Show that the diagonal invertible matrices form an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of order $(p-1)^2$.

   e) Find an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of order p. Make sure to show this is a subgroup.

4) A polygon is regular if all the sides have the same length and all the angles have the same measure. The regular n-gon has n sides. For example, a regular 4-gon is a square. For each $n \geqslant 3$, D_n denotes the group of symmetries of the regular n-gon. Fix a regular n-gon in the Cartesian plane so that its vertices are equidistant from the origin and one lies on the x-axis.

> Theorem: $|D_n| = 2n$.

   (a) The group D_n contains an element r of order n. Describe it.

   (b) For each edge of the n-gon, the group D_n contains a particular element of order 2 that corresponds to some reflection. Similarly, for each vertex of the n-gon, the group D_n contains a particular element of order 2 that corresponds to some reflection. Describe these elements. Explain why these give you a total of n distinct reflections. Hint: you want to distinguish between the cases when n is even or odd.

   (c) Show that D_n can be generated by just one rotation and one reflection.