Homework #6

Problems to hand in on Thursday, March 14, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Let R be a commutative ring. Recall that $e \in R$ is an idempotent if $e^2 = e$.

a) Show that if e is an idempotent, then so is $1 - e$.

> Fact: If $e \neq 0, 1$ is an idempotent, then the ideal generated by e is a ring of its own, with the same multiplication and addition structure as R, but with a *different* multiplicative identity: e. We write Re to represent this ring.

b) Show that the map

$$\begin{aligned} R &\xrightarrow{\varphi} Re \times R(1-e) \\ r &\longrightarrow (re, r - re) \end{aligned}$$

is a ring isomorphism.

c) Show that a ring R is isomorphic to a direct product of two nonzero rings if and only if R contains an idempotent element other than 0_R and 1_R.

Solution.

(a) If $e^2 = e$, then $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$.

(b) We first check the homomorphism conditions. The map takes 1 to $(e, 1-e) = (1_{Re}, 1_{R(1-e)}) = 1_{Re \times R(1-e)}$. We also have

$$\varphi(r+s) = ((r+s)e, (r+s)(1-e)) = (re, r(1-e)) + (se, s(1-e)) = \varphi(r) + \varphi(s)$$

and

$$\varphi(rs) = ((rs)e, (rs)(1-e)) = ((rs)e^2, (rs)(1-e)^2) = \varphi(r)\varphi(s).$$

Now, we check that it is a bijection. If $r \in \ker(\varphi)$, then $re = r(1-e) = 0$, so $r = re + r(1-e) = 0$, so φ is injective. Note that $e(1-e) = e - e^2 = 0$. Now, given $(re, s(1-e)) \in Re \times R(1-e)$ with $r, s \in R$, we have

$$\varphi(re + s(1-e)) = ((re + s(1-e))e, (re + s(1-e))(1-e)) = (re, s(1-e)).$$

Thus, φ is surjective, concluding the proof.

(c) If R has an idempotent $e \neq 0, 1$, then Re and $R(1-e)$ are nonzero rings, and $R \cong Re \times R(1-e)$ by the previous part. Conversely, if $R \cong S \times T$ with $S, T \neq \{0\}$, then the element $(1_S, 0_T) \neq (0_S, 0_T) = 0_{S \times T}$ and $(1_S, 0_T) \neq (1_S, 1_T) = 1_{S \times T}$, and this element is an idempotent.

2) Recall: an ideal $P \neq R$ in a ring R is prime if $ab \in P$ implies $a \in P$ or $b \in P$.

a) Prove that P is prime if and only if R/P is a domain.

b) Use the first isomorphism theorem to show that the ideals (x) and $(2, x)$ in $\mathbb{Z}[x]$ are prime ideals.[1][2]

c) Show that the ideal $(4, x)$ in $\mathbb{Z}[x]$ is not prime.

d) Show that the ideal $(2, \sqrt{10})$ in $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$ is prime.

e) Is the ideal (2) in $\mathbb{Z}[i]$ a prime ideal?

Solution.

(a) Suppose that $P \neq R$ is not prime. Then, there are $a, b \notin P$ with $ab \in P$. The elements $a + P, b + P$ are nonzero in R/P, but $(a + P)(b + P) = ab + P = 0 + P$ is the zero element in R/P, so R/P is not a domain. The converse is similar.

(b) First, we claim that $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. There is a homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ given by evaluation at zero. The kernel of this consists of polynomials with zero constant term, which is just the multiples of x, i.e., (x). This map is surjective, since any integer can be the value-at-zero of an integer polynomial. By the first isomorphism theorem, the claim is shown. Now, since $\mathbb{Z}$ is a domain, by the previous part, (x) is prime.

Second, we claim that $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}_2$. There is a homomorphism $\mathbb{Z}[x] \to \mathbb{Z}_2$ given by evaluation at zero, taken modulo 2. The kernel of this consists of polynomials with even constant term, which we have seen in class is $(2, x)$. This map is surjective, since the value-at-zero of an integer polynomial can be either even or odd. By the first isomorphism theorem, the claim is shown. Now, since $\mathbb{Z}_2$ is a domain, by the previous part, $(2, x)$ is prime.

(c) We need only note that $2 \notin (4, x)$, but $2 \cdot 2 \in (4, x)$.

(d) We claim first that $a + b\sqrt{10} \in (2, \sqrt{10})$ if and only if a is even. Indeed,

$$\begin{aligned}(2, \sqrt{10}) &= \{2(a_1 + b_1\sqrt{10}) + \sqrt{10}(a_2 + b_2\sqrt{10}) \mid a_1, a_2, b_1, b_2 \in \mathbb{Z}\} \\ &= \{(2a_1 + 10b_2) + (2b_1 + a_2)\sqrt{10} \mid a_1, a_2, b_1, b_2 \in \mathbb{Z}\} \\ &= \{2a' + b'\sqrt{10} \mid a', b' \in \mathbb{Z}\}.\end{aligned}$$

Now, given two elements $a + b\sqrt{10}, c + d\sqrt{10}$ *not* in $(2, \sqrt{10})$, we know that a, c are odd. Then,

$$(a + b\sqrt{10})(c + d\sqrt{10}) = (ac + 10bd) + (ad + bc)\sqrt{10}$$

has $ac + 10bd$ odd, so the product is again not in $(2, \sqrt{10})$. We conclude that the ideal is prime.

(e) Not prime! Observe that $(1 + i)(1 - i) = 1 - i^2 = 2$. However, $1 + i$ and $1 - i$ are not multiples of 2 in $\mathbb{Z}[i]$. This needs to be checked; we can check it by computing $2(a + bi) = (2a) + (2b)i$, so the real coefficient of an element in (2) must be even, which is not the case of $1 \pm i$.

[1] Hint: For the first one, consider the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ "evaluate at zero".
[2] Reminder: $(2, x)$ refers to the ideal generated by 2 and x.