Homework #3

Problems to hand in on Thursday, February 7, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Prove that there is no ring homomorphism[1] $\mathbb{Z}_n \longrightarrow \mathbb{Z}$ for any integer $n > 1$.

2) Let R be a ring and S and T subrings of R.

   (a) Prove or disprove: $S \cap T$ is a subring of R.

   (b) Prove or disprove: $S \cup T$ is a subring of R.

3) An element $r \neq 0$ in a commutative ring R is said to be a *zerodivisor* if there exists a nonzero element $s \in R$ such that $rs = 0$.

   (a) Given a nonzero element $r \in R$, prove that r is not a zerodivisor if and only if the map $R \longrightarrow R$ given by multiplication by r, meaning the map $s \mapsto rs$, is injective.

   (b) Describe all the zerodivisors in $\mathbb{Z}_n$ in terms of the prime factorization of n or their greatest common divisor with n.

4) A *unit* u in a ring R is an invertible element, meaning there exists $s \in R$ such that $su = us = 1$.

   (a) Show that if u is a unit in R, then u is not a zerodivisor.

   (b) If $u \neq 0$ is not a zerodivisor in a commutative ring R, does that imply it is a unit?

   (c) Show that if a and b are units, then ab is a unit.

   (d) Prove or disprove: the set of all units in a commutative ring R forms a subring.

   (e) Describe all the units in $\mathbb{Z}_n$ in terms of the prime factorization of n or their greatest common divisor with n.

[1] Reminder: For us, a ring homomorphism must preserve multiplicative identities.