Homework #10

Problems to hand in on Thursday, April 11, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) The **center** of a group G is the set

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

(a) Show that the center of G is an abelian subgroup of G.

(b) Show that the center of G is a normal subgroup of G.

(c) Show that $Z(G) = G$ if and only if G is abelian.

(d) Compute the center of D_4.

(e) Compute the center of S_3.

(f) Compute the center of $GL_2(\mathbb{R})$.

2) Any group G acts on itself by conjugation: $g \cdot h = ghg^{-1}$. The orbits of this action are called **conjugacy classes**.

(a) Show $h \in Z(G)$ if and only h is a fixed point of the conjugation action.

(b) Show a subgroup H of G is normal if and only if it is a disjoint union of conjugacy classes.

(c) Describe the partition of $\mathcal{S}_5$ into its conjugacy classes.

(d) Show that the only nontrivial normal subgroup of $\mathcal{S}_5$ is $\mathcal{A}_5$.[1]

3) Let p be a prime, and G be a finite group with $p \,\big|\, |G|$. Consider the set

$$X = \{(g_1, \dots, g_p) \in \underbrace{G \times \cdots \times G}_{p-\text{times}} \mid g_1 g_2 \cdots g_p = e\}.$$

The group $\mathbb{Z}_p$ acts on X by rotating elements: $[i]_p \cdot (g_1, \dots, g_p) = (g_{1+i}, \dots, g_p, g_1, \dots, g_i)$.

(a) Show that X has $|G|^{p-1}$ elements, so $p \,\big|\, |X|$.

(b) Show that the orbits of the action of $\mathbb{Z}_p$ on X either have 1 or p elements, and the orbits of order 1 are either $(e, e, \dots, e)$ or of the form $(g, g, \dots, g)$ with $|g| = p$.

(c) Show that G contains an element of order p.

[1] Hint: By (b), a normal subgroup is a union of conjugacy classes, one of which is the identity. Use the sizes of these conjugacy classes from (c), plus Lagrange's Theorem, to narrow down the list, and finally show that on your shortlist, the only collection closed under products is $\mathcal{A}_5$.