Homework #1

Problems to hand in on Thursday, January 24, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) In this problem, we will give two proofs of the following FACT:

"*If n is an odd integer, the remainder of n^2 when divided by 8 is 1.*"

(a) First, prove the FACT directly by writing $n = 2k + 1$ using the division algorithm and "FOIL"ing.

(b) Second, show that n is congruent to either $1, 3, 5$, or 7 modulo 8. Show that if the FACT is true when $n = 1, 3, 5$, or 7, then it holds for every odd integer, and complete the proof.

2) Show that if p is a prime integer *other than* ± 2 *or* ± 3, then $p^2 - 1$ is a multiple of 24.

3) Let $f(x)$ and $g(x)$ be two polynomials with integer coefficients. We say that *f is a factor of g in the ring $\mathbb{Z}[x]$* if there is another polynomial with integer coefficients, $h(x)$, such that $g = fh$.

(a) Show that, for any n, $f(x) = x - 1$ is a factor of $g(x) = x^n - 1$ in the ring $\mathbb{Z}[x]$.

(b) Use this to show that any power of 10, $100 \cdots 00$, is congruent to 1 modulo 9.

(c) Use this to show that any positive integer is congruent to the sum of its digits modulo 9.

(d) Now show that if $a = 100 \cdots 00$ has an *even* number of zeroes, then $a \equiv 1 \mod 11$, and if $a = 100 \cdots 00$ has an *odd* number of zeroes, then $a \equiv -1 \mod 11$.

(e) Show that any positive integer n is congruent to

$$(\text{unit digit of } n) - (\text{tens digit of } n) + (\text{hundreds digit of } n) - \cdots \pm \cdots$$

modulo 11.

4) For any integer m, we can use the Fundamental Theorem of Arithmetic to write $m = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ where the p_i's are distinct primes in an (essentially) unique way. The natural number a_i is said to be the multiplicity of the prime p_i in m. [By convention, the multiplicity of p in m is 0 if p does not divide m.]

(a) Let d and n be positive integers. Prove that n is a d-th power of some other integer if and only if for every prime p, the multiplicity of p in n is divisible by d.

(b) Prove that if n is not a d-th power of some other integer, then $\sqrt[d]{n}$ is irrational. [Hint: try proof by contradiction.]

5) Let n and d be non-negative integers. The notation $\binom{n}{m}$ denotes the quantity $\frac{n!}{m!(n-m)!}$. [By convention, we define $0! = 1$.]

(a) Show that for all $1 \leqslant d < n$, $\binom{n}{d} = \binom{n-1}{d} + \binom{n-1}{d-1}$.

(b) Use the previous part to show that $\binom{n}{d}$ is an integer for any $0 \leqslant d \leqslant n$.

(c) Use the fundamental theorem of arithmetic to show that if p is prime and $1 \leqslant d < p$, then $p | \binom{p}{d}$.