Math 412 Adventure sheet on the Orbit Stabilizer Theorem

Fix a group action of the group G on the set X.
DEFINITION: The **orbit** of an element $x \in X$ is the subset of X

$$O(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

DEFINITION: The **stabilizer** of an element $x \in X$ is the subgroup of G

$$\text{Stab}(x) = \{g \in G \mid g(x) = x\} \subset G.$$

ORBIT-STABILIZER THEOREM: *If a finite group G acts on a set X, then for every $x \in X$, we have*

$$|G| = |O(x)| \cdot |\text{Stab}(x)|.$$

A. Let D_4 be the symmetry group of the square. Consider the natural action of D_4 on the square with vertices $(\pm 1, \pm 1)$ by rotations and reflections.

(1) Complete the following chart which records, for different points of the square, the orbit, stabilizer, and cardinalities of each.

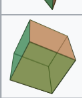
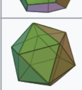
$(x, y) \in \mathbb{R}^2$	$O(x, y)$	stab(x, y)	# $O(x, y)$	# stab(x, y)
$(0, 0)$				
$(1, 0)$				
$(1, 1)$				
$(1, \frac{1}{10})$				
$(\frac{1}{2}, \frac{1}{3})$				

(2) Verify the orbit stabilizer theorem for each of the five points in your chart.

B. THE STABILIZER OF EVERY POINT IS A SUBGROUP. Assume a group G acts on a set X. Let $x \in X$.

(1) Prove that the stabilizer of x is a **subgroup** of G.
(2) Use the Orbit-Stabilizer theorem to prove that the cardinality of every orbit divides $|G|$.
(3) Let G be a group of order 17 and let X be a set with 16 elements. Explain why there is no nontrivial action of G on X. [The trivial action is the one in which $g \cdot x = x$ for all $g \in G$ and all $x \in X$.]

C. SYMMETRY GROUPS OF PLATONIC SOLIDS. There are exactly five convex regular solid figures in $\mathbb{R}^3$.

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Each is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. The chart describes each of these platonic solids. Each platonic solid has a symmetry group which acts naturally on the solid. In particular, each symmetry group also acts on the set of vertices, the set of edges and the set of faces, of the corresponding solid. By analyzing these three actions, we can better understand the symmetry group of each solid.

For each of the 5 platonic solids, complete the following chart:

Action	# orbit	# stab	$\lvert G\rvert$
on Faces			
on edges			
on vertices			

For each of the three actions, does it matter which point $x \in X$ (i.e., which face, edge, or vertex) you use to compute the orbit? Why is the order of the stabilizer the same for each $x \in X$ in each of the three actions? Is this true in general for a group acting on a set? What is special in this case?

D. Consider the group Cube of symmetries of the cube.

(1) Observe that Cube acts on the set of 4 diagonals (from one vertex to its opposite) of the cube.
(2) Show that this action is faithful.[1]
(3) Show that Cube is isomorphic to $\mathcal{S}_4$.[2]
(4) Conclude that the orders of the elements in Cube are exactly $1, 2, 3, 4$, and that Cube is generated by two elements.

E. THE PROOF OF THE ORBIT-STABILIZER THEOREM: Let G act on X. Fix a point $x \in X$.

(1) Show that there is a surjective map of sets $G \to O(x)$ sending each $g \in G$ to $g \cdot x$.
(2) Show that g and h have the same image under this map if and only if $g^{-1}h \in \mathrm{Stab}(x)$.
(3) For each $g \cdot x \in O(x)$, show that the set of elements in G mapping to $g \cdot x$ is the left coset gK where $K = \mathrm{Stab}(x)$.
(4) Show that this map induces a bijection between the set $G/\mathrm{Stab}(x)$ of left cosets of $\mathrm{Stab}(x)$ in G and the orbit $O(x)$.
(5) Prove the Orbit Stabilizer Theorem.

F. LINEAR ACTIONS. Consider the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$ by matrix multiplication (where elements of $\mathbb{R}^2$ are written as columns).

(1) Describe the action in mathematical symbols and prove it is really an action.
(2) What is the stabilizer of the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$?
(3) What is the orbit of the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

G. SYMMETRY GROUPS OF PLATONIC SOLIDS AGAIN.

(1) Show that the symmetry group of the tetraheadron is isomorphic to $\mathcal{A}_4$.
(2) Show that the symmetry group of the octaheadron is isomorphic to $\mathcal{S}_4$.
(3) Can you compute the symmetry groups of the dodecahedron and the icosahedron?

H. REPRESENTATIONS OF A GROUP. A REPRESENTATION of a group G on a vector space V is a group homomorphism $G \longrightarrow GL(V)$ to the set of bijective linear transformations $V \longrightarrow V$.

(1) Explain why a group representation is equivalent to an action of G on V where for each $g \in G$, $\mathrm{ad}(g)$ is a linear transformation.
(2) Show that a group representation is a faithful action if and only if $G \longrightarrow GL(V)$ is injective.
(3) Give an example of a representation of D_n on $\mathbb{R}^2$.

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[1] Hint: Label the diagonals as $1, 2, 3, 4$. Note that every face has one vertex on each diagonal. For each face, list the diagonal of each vertex, conterclockwise, starting with 1. Note that each face has a different list.

[2] Hint: Use the homomorphism $\mathrm{ad} : G \to \mathrm{Bij}(X)$ from the last worksheet.