Math 412: Adventure sheet on groups

DEFINITION: A **group** is a nonempty set G with an operation $\star$ that satisfies the axioms

- Composition is associative: For all $g_1, g_2, g_3 \in G$, we have $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$;
- There is an identity: There exists $e \in G$ such that for all $g \in G$, we have $g \star e = e \star g = g$;
- Every element has an inverse: For all $g \in G$, there exists $h \in G$ such that $g \star h = h \star g = e$.

If we want to specify the operation, we may write $(G, \star)$.

We often just write gh for $g \star h$, and g^{-1} for the inverse of g.

DEFINITION: An **abelian group** is a group $(G, \star)$ with one additional axiom

- For all $g_1, g_2 \in G$, we have $(g_1 \star g_2) = (g_2 \star g_1)$ ($\star$ is commutative).

DEFINITION: A **subgroup** of a group $(G, \star)$ is a subset H that is itself a group under $\star$.

DEFINITION: An element g of a group $(G, \star)$ has **order** n if n is the smallest natural number such that $g^n = g \star g \star \cdots \star g \ (n\,times) = e$. If no such n exists, we say that g has infinite order.

A. SYMMETRY GROUP D_3 OF AN EQUILATERAL TRIANGLE.

There are six different *ways to move an equilateral triangle around and put it in the same spot*. For concreteness, let us assume the triangle has one side horizontal.

k: "keeps put"
r_{120}: rotate $120°$ counterclockwise
r_{240}: rotate $240°$ counterclockwise
f_1: flip the triangle around the vertical axis of symmetry (perpendicular to the horizontal side).
f_2: flip the triangle around the axis of symmetry that is $60°$ counterclockwise from the vertical.
f_3: flip the triangle around the axis of symmetry that is $60°$ clockwise from the vertical.

(1) If you compose any two of these *ways to move an equilateral triangle around and put it in the same spot*, you get another *way to move an equilateral triangle around and put it in the same spot*, which must be one of the six things on the list. Make a table for the operation of composition of these six "rigid motions" of the triangle. Use care with order of operations: the convention should agree with our conventions on reading the table, namely the column corresponds to the first transformation applied, and row corresponds to the second.

	k	r_{120}	r_{240}	f_1	f_2	f_3
k						
r_{120}						
r_{240}						
f_1						
f_2						
f_3						

(2) Explain why the set D_3 of symmetries of an equilateral triangle forms a group, where the operation $\star$ is composition. Be sure to clearly identify the identity element, and the inverse of each element.

(3) Find the order of each element.

B. Consider a set of three objects, labelled x_1, x_2, x_3. Consider the following shuffles of them:

n: no shuffling occurred
s_{12}: swap the first and second items
s_{13}: swap the first and third items
s_{23}: swap the second and third items
m_1: move the last item to the front
m_2: move the front item to the end

(1) Composing two shuffles produces a shuffle. Make a table for composition of shuffles.

	n	s_{12}	s_{13}	s_{23}	m_1	m_2
n						
s_{12}						
s_{13}						
s_{23}						
m_1						
m_2						

(2) The set of all shuffles of three objects forms a group S_3 under composition. What is the identity element S_3? What is the inverse of each element? Find the orders of all elements of S_3.

C. EASY PROOFS: Let $(G, \star)$ be a group.

(1) Prove that the identity element of G is unique.
(2) Fix $g \in G$. Prove that the inverse, g^{-1} of g is unique.
(3) For $a, b \in G$, show that there is exactly one element $x \in G$ such that $a \star x = b$, and exactly one element $y \in G$ such that $y \star a = b$.
(4) Think about what (3) says about the rows and columns of the table of a (finite) group. Why do we call this the **Sudoku Rule**?

D. COMPARING GROUPS

(1) Are either of the groups D_3 or S_3 abelian? How do you know from the tables?
(2) Show that $\{k, r_{120}, r_{240}\}$ is a subgroup of D_3. Show that $\{n, m_1, m_2\}$ is a subgroup of S_3.
(3) What do you think an **isomorphism** of groups should be? Are D_3 and S_3 isomorphic? How could you arrange the tables to make an isomorphism easier to see?

E. Let $GL_2(\mathbb{R})$ be the set of 2×2 invertible matrices with $\mathbb{R}$ entries. Use basic properties of matrices to prove $GL_2(\mathbb{R})$ is a group. Is it abelian? Find an element of order two, an element of order 4, and an element of infinite order. Find an abelian subgroup.

F. Is $\mathbb{Z}_{24}$ a group under addition? What is the identity? What is the inverse of $[a]_{24}$? Is it abelian? What is the order of $[4]_{24}$? Show that the ideal of $\mathbb{Z}_{24}$ generated by $[4]_{24}$ is a subgroup of $(\mathbb{Z}_{24}, +)$. Find a subgroup with two elements. Find a subgroup with 3 elements.

G. Explain why $\mathbb{Z}_{24}$ is NOT a group under multiplication. Explain why the subset of units $\mathbb{Z}_{24}^{\times}$ in $\mathbb{Z}_{24}$ is a group under multiplication. Is it abelian? What is the identity? Find the inverse of each element in the group $(\mathbb{Z}_{24}, \times)$. Find the order of each element.

H. For which of the following pairs $(G, \cdot)$, where G is a set and $\cdot$ is an operation on G, which ones are groups?

(1) $(\mathbb{N}, +)$.
(2) $(\mathbb{N}, \times)$.
(3) $(\mathbb{Z}, +)$.
(4) $(\mathbb{Z}, \times)$.
(5) $(\mathbb{R}_{\geqslant 0}, +)$.
(6) $(\mathbb{R}, +)$.
(7) $(\mathbb{R}_{\geqslant 0}, \times)$.
(8) $(\mathbb{R}, \times)$.
(9) $(\mathbb{R} \setminus \{0\}, \times)$.
(10) $(\mathbb{Z}_n, +)$.
(11) $(\mathbb{Z}_n, \times)$.
(12) $(\mathbb{Z}_n \setminus \{0\}, \times)$.

(13) $(\mathrm{GL}_2(\mathbb{R}), \times)$, where $\mathrm{GL}_2(\mathbb{R})$ is the set of 2×2 invertible matrices with entries in $\mathbb{R}$.
(14) $(\mathrm{SL}_2(\mathbb{R}), \times)$, where $\mathrm{SL}_2(\mathbb{R})$ is the set of 2×2 matrices with entries in $\mathbb{R}$ and determinant 1.