Math 412 Adventure sheet on group actions

DEFINITION: Let $(G, \star)$ be a group. Let X be a set. A **group action** of G on X is a function

$$G \times X \to X \qquad\qquad (g, x) \to g \cdot x,$$

satisfying the axioms:

(1) $e_G \cdot x = x$ for all $x \in X$, and
(2) $h \cdot (g \cdot x) = (h \star g) \cdot x$ for all $g, h \in G$ and all $x \in X$.

DEFINITION: A group action of the group G on the set X is **faithful** if the only element $g \in G$ such that $g \cdot x = x$ for all $x \in X$ is the identity.

DEFINITION: Fix a group action of the group G on the set X. The **orbit** of an element $x \in X$ is the subset of X

$$O(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

A. Let D_4 be the symmetry group of the square. The group D_4 acts on the set of points X of the square in a canonical way. Note that X is an infinite set of points.

(1) Draw a picture of the square in the Cartesian plane so its vertices are $(\pm 1, \pm 1)$. Explain the canonical action of D_4 on the square.
(2) Compute the orbit of each the following types of points (and sketch): the origin, a vertex, a nonzero point on a diagonal of the square, a nonzero point on the horizontal axis of symmetry, a nonzero point not on any axis of symmetry.
(3) What is the largest number of points any orbit can have? Find an explicit point whose orbit achieves this value.
(4) True or False: under the given action of D_4 on the square, all vertices have the same orbit.
(5) True or False: the given action of D_4 on the square is faithful.

Solution.

(1) The action is by reflections and rotations, as usual.
(2) origin: just itself; a vertex: all vertices; nonzero point on diagonal: consists of one point on each diagonal; a nonzero point on the horizontal axis of symmetry: two points on the vertical axis and two points on the vertical axis; a point on no axis of symmetry: eight points.
(3) Eight: a point on no axis of symmetry.
(4) True.
(5) True.

B. Let $\mathcal{S}_4$ be the group of permutations of the set $\{1, 2, 3, 4\}$. There is a canonical action of $\mathcal{S}_4$ on the set $X = \{1, 2, 3, 4\}$ defined by $g \cdot x = g(x)$.

(1) Verify that this is a group action.
(2) Find the orbit of the element $4 \in X$. Are any two elements of X in the same orbit?
(3) Let Y be the set of *subsets* of $\{1, 2, 3, 4\}$. Describe a natural action of $\mathcal{S}_4$ taking every subset of $\{1, 2, 3, 4\}$ to another. Quickly convince yourself that your action satisfies the axioms of an action.
(4) Find the orbit of the set $\{1\} \in Y$ under the action of S_4 you described in (3). What is the cardinality of this orbit?
(5) Find the orbit of $\{1, 2\} \in Y$. What is the cardinality of this orbit?

(6) A **fixed point** is a point $x \in Y$ such that $g \cdot x = x$ for all $g \in G$. Does Y have any fixed points under this action?
(7) The action of S_4 on Y partitions Y up into disjoint orbits. Describe these.

Solution.

(1) Follows from the fact that the group rule is just composition of functions.
(2) The orbit is the whole set $\{1, 2, 3, 4\}$.
(3) For a subset $S \subseteq \{1, 2, 3, 4\}$, let the group act by acting on the elements of the set: $\sigma(S) = \{\sigma(s) \mid s \in S\}$. The identity fixes S, and $\sigma\tau(S) = \sigma(\tau(S))$.
(4) $O(\{1\}) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.
(5) $O(\{1, 2\}) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.
(6) Yes: $\varnothing$ and $\{1, 2, 3, 4\}$ are the two fixed points.
(7) The orbits are:
  (a) $\{\varnothing\}$
  (b) $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
  (c) $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$
  (d) $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$
  (e) $\{\{1, 2, 3, 4\}\}$.

C. Fix a group action of a group G on a set X.

(1) Observe that if you fix an element $g \in G$, then the rule $x \to g \cdot x$ is a function from $X \to X$. We denote this function as $\mathrm{ad}(g) : X \to X$.
(2) Verify that $\mathrm{ad}(e)$ is the identity function on X.
(3) Show that, for $g, h \in G$, $\mathrm{ad}(g) \circ \mathrm{ad}(h) = \mathrm{ad}(gh)$.
(4) Show that ad is a group homomorphism from G to $\mathrm{Bij}(X)$, the group of bijections of X (under composition).
(5) Show that ad is injective if and only if the action is faithful.

Solution.

(1) OK!
(2) $\mathrm{ad}(e)(x) = e \cdot x = x$ for all $x \in X$ by axiom (1).
(3) $(\mathrm{ad}(g) \circ \mathrm{ad}(h))(x) = \mathrm{ad}(g)(\mathrm{ad}(h)(x)) = g \cdot (h \cdot x) = gh \cdot x = \mathrm{ad}(gh)(x)$ for all $x \in X$, so $\mathrm{ad}(h) = \mathrm{ad}(gh)$.
(4) We observe that for any $g \in G$, $\mathrm{ad}(g) : X \to X$ is a bijection, since it has an inverse, namely, $\mathrm{ad}(g^{-1})$. Now, the previous part shows that this is a homomorphism.
(5) The kernel of ad is the set of group elements that map to the identity function. We note that $\mathrm{ad}(g) = \mathrm{id}_X$ means that $g \cdot x = x$ for all $x \in X$. Thus, $\ker(\mathrm{ad}) = \{e\}$ if and only if the action is faithful.

D. CAYLEY'S THEOREM. Let G be a finite group of order n.

(1) Show that the rule $g \cdot x = gx$ defines a group action of G on itself ($X = G$).
(2) Show that this action is faithful.
(3) Conclude that G is isomorphic to a subgroup of $\mathcal{S}_n$.

Solution.

(1) Both axioms are immediate.
(2) We need to see that if $gx = x$ for all $x \in G$, then $g = e$. This is clear, since $g = gxx^{-1} = xx^{-1} = e$.
(3) By the previous problem, there is an injective homomorphism from $G \to \mathrm{Bij}(G)$, so G is isomorphic to a subgroup of $\mathrm{Bij}(G)$. Note that $\mathrm{Bij}(G) \cong \mathcal{S}_n$. Thus, G is isomorphic to a subgroup of $\mathcal{S}_n$.

E. Consider the group Cube of symmetries of the cube. Recall that $|\mathrm{Cube}| = 24$.

(1) Observe that Cube acts on the set of diagonals (from one vertex to its opposite) of the cube.
(2) Show that this action is faithful.[1]
(3) Show that Cube is isomorphic to $\mathcal{S}_4$.
(4) Conclude that the orders of the elements in Cube are exactly $1, 2, 3, 4$, and that Cube is generated by two elements.

Solution.

(1) A symmetry must take a diagonal to a diagonal; it is clear that this is compatible with composition.
(2) Following the hint, once we label the diagonals, every face is determined by the order in which the diagonals meet its vertices. Thus, if an element of the group fixes all four diagonals, then it fixes all of the faces, so it can only be the identity.
(3) By the last part, we obtain an injective homomorphism from Cube to $\mathcal{S}_4$. Since these groups have the same order, this map must be bijective.
(4) This follows from the fact that these statements hold in $\mathcal{S}_4$.

F. There can be different actions of the same group G on the same set X. For example, the group $\mathbb{Z}_2$ can act on the Cartesian plane $\mathbb{R}^2$ as follows:

$$[0]_2 \cdot (x, y) = (x, y) \qquad [1]_2 \cdot (x, y) = (y, x).$$

A *different* group action is as follows:

$$[0]_2 \cdot (x, y) = (x, y) \qquad [1]_2 \cdot (x, y) = (-x, -y).$$

(1) Verify that these are both group actions. Describe them geometrically.
(2) Find another group action, different from these, of $\mathbb{Z}_2$ on $\mathbb{R}^2$. There are many possibilities; check on a neighboring group to see what they came up with as well.
(3) For each of the three actions in play here, describe the orbits. How many elements can be in an orbit?

Solution.

(1) Remembering that the operation in $\mathbb{Z}_2$ is addition, we only need to check that $g_1 \cdot (g_2 \cdot (x, y)) = (g_1 + g_2) \cdot (x, y)$. for all $g_1, g_2 \in \mathbb{Z}_2$ and all $(x, y) \in \mathbb{R}^2$. Since $\mathbb{Z}_2$ has only the two elements $[0]$ and $[1]$, and we know that $[0]$ does nothing to any point (x, y) in $\mathbb{R}^2$, we only need to check that $g_1 \cdot (g_2 \cdot (x, y)) = (g_1 + g_2) \cdot (x, y)$ for $g_1 = g_2 = [1]$. (You should double check the others too!). Since $[1] + [1] = [0]$, we have $[1] \cdot ([1] \cdot (x, y)) = [1] \cdot (y, x) = (x, y) = [0] \cdot (x, y)$ as needed to

[1] Hint: Label the diagonals as $1, 2, 3, 4$. Note that every face has one vertex on each diagonal. For each face, list the diagonal of each vertex, conterclockwise, starting with 1. Note that each face has a different list.

verify the first mapping is an action. For the second, we have $[1] \cdot ([1] \cdot (x; y)) = [1] \cdot (-x, -y) = (x, y) = [0] \cdot (x, y)$. QED.

(2) One example is given by $[1] \cdot (x, y) = (-x, y)$.

(3) The orbits of the first action are either of cardinality two: $\{(a, b), (b, a)\}$, or cardinality one: $\{(a, a)\}$. The orbits of the second action are all of cardinality two: $\{(a, b), (-a, -b)\}$ except for the origin, which is its own orbit $\{(0, 0)\}$. The orbits of the third action are cardinality two: $\{(a, b), (-a, b)\}$, except for points on the y-axis which have orbits of cardinality one: $\{(0, b)\}$.

G. Let a group G act on a set X.

(1) If G has n elements, explain why every orbit has at most n elements.
(2) If X has m elements, explain why every orbit has at most m elements.
(3) Prove that the relation "$x \sim y$ if $x \in O(y)$" is an equivalence relation on X.
(4) Prove that the orbit of x and the orbit of y either coincide exactly or are disjoint.

Solution.

(1)
- Reflexive: $x \sim x$ for all $x \in X$ because for $e_G \in G$, we have $e_G \cdot x = x$ by the first axiom of group
- Symmetric: Say $x \sim y$. This means that there exists $g \in G$ such that $x = g \cdot y$. But then we can apply $g^{-1} \in G$ to x to get $g^{-1} \cdot x = g^{-1} \cdot (g \cdot y) = (g^{-1}g) \cdot y = e_G \cdot y = y$ where we have used both the first and second axioms of group actions for the penultimate and final equalities above. Thus $y = g^{-1} \cdot x$, and so $y \sim x$. So symmetry holds for $\sim$.
- Transitivity: Assume $x \sim y$ and $y \sim z$. We need to show $x \sim z$. By definition, there exists g, h such that $x = g \cdot y$ and $y = h \cdot z$. Substituting: $x = g \cdot y = g \cdot (h \cdot z)$. By the second group action axiom, $x = (gh) \cdot z$, so also $x \sim z$. Transitivity holds. This shows that $\sim$ is an equivalence relation.

(2) Say that $O(x) \smallsetminus O(y)$ is not empty. Take $z \in O(x) \smallsetminus O(y)$. We can write $z = g \cdot x = h \cdot y$ for some $g, h \in G$. Apply g^{-1} to both sides: $g^{-1}(g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x = g^{-1}h \cdot x \in O(y)$: This shows that $O(x) \subseteq O(y)$, and a similar argument shows the reverse inclusion. So $O(x) = O(y)$ if they have any point at all in common.

(3) Every point is in some orbit (since $x \in O(x)$). This means $X = \bigcup_{x \in X} O(x)$. Now, just throw away any orbits that are doubled up. The remaining orbits are all disjoint.