Math 412
Final Exam review problems

True or false. Justify!

1) $D_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.
2) $D_3 \cong \mathcal{S}_3$.
3) $D_4 \cong \mathcal{S}_4$.
4) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$.
5) $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$.
6) An element g of a group G can satisfy $g^{30} = e$ and have order 6.
7) An element g of a group G can satisfy $g^{30} = e$ and have order 7.
8) Every normal subgroup is the kernel of some group homomorphism.
9) If a group G contains an element of infinite order, then G is infinite.
10) If a group G contains a nontrivial element of finite order, then G is finite.
11) If every element in a group G has finite order, then G is finite.
12) There are no nontrivial simple abelian groups.
13) There exists a group G of order 8 acting on a set X such that some $x \in X$ has orbit of cardinality 5.
14) There exists a group G of order 12 acting on a set X such that some $x \in X$ has orbit of cardinality 6 and stabilizer of order 4.
15) When a group G acts on a set X, a point $x \in X$ is fixed if and only if Stab$(x) = G$.
16) For every $n \geqslant 5$, $\mathcal{A}_n$ is simple.
17) The group of units in $\mathbb{Z}$ is cyclic.
18) The group of units in $\mathbb{Z}_{51}$ is cyclic.
19) The group of units in $\mathbb{Z}_8$ is cyclic.
20) The subgroup of $SL_2(\mathbb{R})$ generated by $\begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$ is isomorphic to $\mathbb{Z}_n$.
21) There always exists a group homomorphism between any two groups.
22) If H is an abelian subgroup of the (possibly nonabelian) group G, then H is normal.
23) If H is a subgroup of an abelian group G, then G/H is abelian.
24) If a group G of order 14 acts on a set with 14 elements, it's possible the total number of orbits is 3.
25) When the Klein-4 group acts on a set of 11 elements, there are at most 4 orbits.
26) The center of an abelian group G is the set of all elements of G.
27) The center of a group G is an abelian group.
28) The center of a group is always a normal subgroup.
29) If every proper subgroup of a group G is cyclic, then G is cyclic.
30) There exists a surjective group homomorphism $\mathbb{Z}_7 \longrightarrow \mathbb{Z}_5$.
31) There exists an injective group homomorphism $\mathcal{S}_7 \longrightarrow \mathcal{S}_8$.
32) There exists an injective group homomorphism $\mathbb{Z}_7 \longrightarrow \mathbb{Z}_8$.
33) When a nontrivial group acts on itself by conjugation, there is always a fixed point.
34) When a nontrivial group acts on itself by left multiplication, there is always a fixed point.
35) If an action of the group G on the set X has at least one fixed point, then the action is faithful.
36) If an action of the group G on the set X has no fixed points, the action is faithful.
37) The quotient group G/K is a subset of G.
38) The elements gK and hK are equal in G/K if and only if $g = h$.
39) If G is a group of order n and $k|n$, there is an element of G of order k.
40) If G is a group of order n and $k|n$, there is a subgroup of G of order k.

41) Every element in $\mathcal{S}_{123}$ is a product of elements of order 2.
42) Every element in $\mathcal{S}_{123}$ is a product of elements of order 3.
43) There exists a field $\mathbb{F}$ such that both $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \times)$ are cyclic.
44) Every quotient of a nonabelian group is nonabelian.
45) A subgroup of order 2 is always normal.
46) Every subgroup of index 2 is normal.
47) Every subgroup of index 3 is normal.
48) For a ring homomorphism $f\colon R \to S$, $f^\times\colon R^\times \to S^\times$ given by $f^\times(u) = f(u)$ is a group homomorphism.
49) The image if a group homomorphism $G \longrightarrow H$ with G abelian is always an abelian subgroup of H.
50) If there exists a nontrivial group homomorphism $G \longrightarrow H$ with G abelian, then H is abelian.
51) Suppose that G acts on itself by conjugation. Then not every point is a fixed point.
52) The quotient group $\mathbb{Q}/\mathbb{Z}$ is a finite group.
53) Let $\mathbb{F}$ be a finite field with a nonidentity element g satisfying $g^2 = 1$. Then $|\mathbb{F}|$ is odd.
54) There exists a surjective group homomorphism $\mathcal{S}_5 \longrightarrow \mathbb{Z}_3$.
55) If $g^{18} = e$, then the order of g is 18.
56) A group of order 400 can have an element of order 19.
57) The intersection of two normal subgroups is a normal subgroup.
58) If R is a finite ring, $R^\times$ is a cyclic group.
59) Given any ring $(R, +, \times)$, $(R, +)$ is always a group.
60) Given any ring $(R, +, \times)$, $(R, \times)$ is always a group.
61) The rings $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^2+1)$ are isomorphic.
62) $\mathcal{S}_3$ is a cyclic group.
63) There are two nonisomorphic cyclic groups of order 20.
64) There exists a subgroup of S_5 that is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.
65) Every 4-cycle in $\mathcal{S}_{103}$ is odd.
66) $\mathcal{S}_{120}$ has no subgroup isomorphic to D_{60}.
67) Every group of order 12 contains an element of order 4.
68) Every group of order 120 contains an element of order 3.
69) Let R be the subgroup of all rotations in D_4. Then $D_4/R \cong \mathbb{Z}_3^\times$.
70) Given a group G and $x \in G$, x defines a group homomorphism $G \longrightarrow G$ by $g \mapsto gxg^{-1}$.
71) If F is a field and R is a nonzero ring, every ring homomorphism $F \longrightarrow R$ is injective.
72) If G and H are two groups of the same order, then $G \cong H$.
73) Every group of order 29 is simple.
74) The image of a group homomorphism is always a normal subgroup.
75) The kernel of a group homomorphism is always a normal subgroup.
76) Every finite group G is isomorphic to a subgroup of $\mathcal{S}_n$ for some n.
77) Every quotient of a domain is a domain.
78) Every quotient of a field is a field.
79) In any group G, the product of elements of finite order always has finite order.
80) Every nontrivial group has at least two subgroups.
81) Every nontrivial group has at least two normal subgroups.
82) Every ring homomorphism $M_2(\mathbb{R}) \to R$ to a nontrivial ring R is injective.
83) There are exactly 2 ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$.
84) Every subgroup of an abelian group is abelian.
85) There are no group homomorphisms $\mathbb{Z}_2 \to \mathbb{Z}_4$.
86) There are no group homomorphisms $\mathbb{Z}_n \to \mathbb{Z}$.
87) In $\mathbb{Z}$, if $n = p_1 \cdots p_t = q_1 \cdots q_s$, for primes p_i, q_j, then $s = t$ and $p_1 = q_1, \ldots, p_s = q_s$.
88) In general, the fastest way to find the gcd of two large integers is to factor them into primes.
89) The equation $[a]_n\, x = [b]_n$ has a solution in $\mathbb{Z}_n$ if and only if $\gcd(a, n) = 1$.

90) The system of equations $7|(x+3)$ and $11|(x-1)$ has a solution modulo 77.
91) The system of equations $3|x$ and $6|(x-1)$ has a solution modulo 18.
92) If $n|a$ and $m|a$, then $nm|a$.
93) Given any ring R, there exists exactly one ring homomorphism $\mathbb{Z} \longrightarrow R$.
94) Given any ring R, there exists exactly one ring homomorphism $R \longrightarrow \mathbb{Z}$.
95) Given any ring R, there exists exactly one ring homomorphism $\mathbb{Z}_n \longrightarrow R$.
96) Given any ring R, there exists exactly one ring homomorphism $R \longrightarrow \mathbb{Z}_n$.
97) Every element in $\mathbb{Z}$ is a unit.
98) The additive inverse of $[5]_{77}$ in $\mathbb{Z}_{77}$ is $[149]_{77}$.
99) The multiplicative inverse of $[5]_{77}$ in $\mathbb{Z}_{77}$ is $[108]_{77}$.
100) Every nonzero ring contains at least two ideals.
101) Every domain is a field.
102) Every field is a domain.
103) The zero ring is a domain.
104) There always exists a ring homomorphism between any two rings.
105) Any commutative ring that has only two ideals is a field.
106) The kernel of any ring homomorphism is an ideal.
107) The kernel of any ring homomorphism is a subring.
108) The image of any ring homomorphism is an ideal.
109) The image of any ring homomorphism is a subring.
110) If R is a commutative ring and $(g) = R$, then g is a unit.
111) If R is a domain, then $R[x]$ is a domain.
112) If F is a field, then $F[x]$ is a field.
113) Every reducible polynomial of degree 4 in $F[x]$ for a field F has a root in F.
114) Every reducible polynomial of degree 3 in $F[x]$ for a field F has a root in F.
115) If $p(x) \in \mathbb{Z}_2[x]$ has degree 3, then $\mathbb{Z}_2[x]/(p(x))$ has 4 elements.
116) If $p(x) \in F[x]$ for some field F is irreducible, then gcd$(p(x), f(x))$ is 1 or p.
117) If F is a field, the remainder of dividing $f(x)$ by $x - a$ is $f(a)$.
118) Modern algebra is fun!
119) The ring $\mathbb{Z}_n[x]$ is a domain.
120) $\mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_{60}$ as rings.
121) $\mathbb{Z}_{10} \times \mathbb{Z}_6 \cong \mathbb{Z}_{60}$ as rings.
122) If f and g differ by a unit in $F[x]$, where F is a field, then $(f, g) = 1$.
123) If $uf + vg = 4$ in $\mathbb{Q}[x]$, then $f + (g)$ is a unit in $\mathbb{Q}[x]/(g)$.
124) In $R[x]$, the product of two monic polynomials can be zero.
125) If F is a field, the map $F[x] \longrightarrow F$ sending each polynomial to its constant term is a ring homomorphism.
126) $x^3 + 2$ is a unit in $\mathbb{Z}_5[x]/(x^4 - x^2)$.
127) The quotient ring $\mathbb{R}[x]/(x^3 - x - 6)$ is a field.
128) Every ideal is the kernel of some ring homomorphism.
129) Any subring of a domain is a domain.
130) Any subring of a field is a field.
131) $2^3 \equiv 2^7 \mod 5$.
132) Every integer is congruent to the sum of its digits modulo 11.
133) An element of a commutative ring R cannot be both a unit and a zerodivisor.
134) A subset of a ring that is also a ring is a subring.
135) $\mathbb{Z}_n$ is a domain if and only if it is a field.
136) If $ua + vb = n$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = n$.
137) If $ua + vb = 1$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = 1$.
138) Every element in $\mathbb{Z}_{11}$ is invertible.

139) In $\mathbb{Z}_{77}$, $(a) = (b)$ if and only if $a = b$.
140) Every ideal in $\mathbb{Z}_{123}$ is principal.
141) In $\mathbb{Z}[x]$, $(a, b) = (\text{gcd}(a, b))$.
142) If R and S are domains, then $R \times S$ is a domain.
143) In any ring R, $ab = 0$ implies $a = 0$ or $b = 0$.
144) In any ring R, we can cancel addition.
145) In any ring R, we can cancel multiplication.
146) On the set of real numbers, $r \sim s$ if and only if $|r| = |s|$ defines an equivalence relation.
147) If a is even and b is odd, (a, b) is even.
148) If $a|b$ and $b|c$, then $a|c$.
149) If I and J are ideals in a ring R, $I \cup J$ is an ideal in R.
150) There is a bijection between ideals in R containing I and ideals in R/I.
151) Every prime ideal is maximal.
152) If I is a prime ideal, then R/I is a field.
153) If R is a field, then R has at most two ideals.
154) If p is a prime and a is any integer, $a^{p-1} \equiv 1 \mod p$.
155) If p is a prime and a is any integer, $a^p \equiv a \mod p$.