Math 412. Adventure sheet on §2.1: Congruence in $\mathbb{Z}$.

DEFINITION: Fix a nonzero integer N. We say that $a, b \in \mathbb{Z}$ are **congruent modulo** N if $N|(a-b)$. We write $a \equiv b \mod N$ for "a is congruent to b modulo N." Parse this notation as $a \equiv b \mod N$: the a and b are the two inputs, and $\equiv \quad \mod N$ is one piece, like a complicated equals sign.

DEFINITION: Fix a nonzero integer N. For $a \in \mathbb{Z}$, the **congruence class of** a **modulo** N is the subset of $\mathbb{Z}$ consisting of all integers congruent to a modulo N; That is, the **congruence class of** a **modulo** N is

$$[a]_N := \{b \in \mathbb{Z} \,|\, b \equiv a \mod N\}.$$

Note here that $[a]_N$ is the **notation** for this congruence class— in particular, $[a]_N$ stands for a *subset of* $\mathbb{Z}$, not a number.

A. WARM-UP: True or False. Justify.

(1) T or F: $5 \equiv 19 \mod 7$,
(2) T or F: $-5 \equiv 20 \mod 10$,
(3) T or F: $-11 \equiv -26 \mod 5$,
(4) T or F: Any two odd integers are congruent modulo 2.
(5) T or F: Any two odd integers are congruent modulo 3.

B. EASY PROOFS.

(1) Show that Congruence Modulo N is an *Equivalence Relation*. That is, prove that
  (a) $a \equiv a \mod N$ (congruence is reflexive);
  (b) If $a \equiv b \mod N$, then $b \equiv a \mod N$ (congruence is symmetric);
  (c) If $a \equiv b \mod N$ and $b \equiv c \mod N$, then $a \equiv c \mod N$ (congruence is transitive).
(2) For a fixed $N > 0$, prove that every $a \in \mathbb{Z}$ is congruent mod N to some $r \in \mathbb{Z}$ such that $0 \leq r < N$.[1]

C. CONGRUENCE CLASS BASICS.

(1) List out (with the help of some "...''s) all of the elements in $[11]_4$.
(2) Given two congruence classes, $[a]_N$ and $[b]_N$, show that[2]

$$\text{either } [a]_N = [b]_N \text{ or } [a]_N \cap [b]_N = \emptyset.$$

(3) Explain why there are exactly N equivalence classes modulo N.
(4) Discuss with your team the following important idea: *Congruence Classes Mod N partition the integers into exactly N nonoverlapping subsets of $\mathbb{Z}$.* Have we proven this? What are these sets when $N = 2$? Can you find a nice way to list out these N sets using the notation $[a]_N$ in general? How does it look in set-builder notation?

D. TRUE OR FALSE? JUSTIFY.

(1) $47 \in [17]_5$.
(2) $[17]_7 \cap [23]_7 = \emptyset$.
(3) $[17]_6 \cap [19]_7 = \emptyset$.
(4) For all integers a, $[a]_{60} \subset [a]_{10}$.

[1] Hint: Division algorithm!

[2] Hint: One form of the contrapositive statement is: if $[a]_N \cap [b]_N \neq \emptyset$, then $[a]_N = [b]_N$. There are standard techniques you know from 217 to show two sets are the same.

E. FUNCTIONS / OPERATIONS ON CONGRUENCE CLASSES.

(1) Take a second to recall the definition of a function. What makes a rule for turning inputs into outputs a well-defined function?

(2) Consider the following rule to turn congruence classes modulo 7 into congruence classes modulo 7:

$$[a]_7 \mapsto [\text{“round down } a \text{ to the nearest multiple of 10”}]_7.$$

Explain carefully why this is *not* a function from congruence classes modulo 7 to congruence classes modulo 7.

(3) Consider the following different rule to turn congruence classes modulo 7 to congruence classes modulo 7:

$$[a]_7 \mapsto [-a]_7.$$

Explain why this *is* a function from congruence classes modulo 7 to congruence classes modulo 7. Explain why this justifies that “taking negatives” is a well-defined function from congruence classes modulo 7 to itself.

F. ADDING & MULTIPLYING CONGRUENCE CLASSES. Fix $N \neq 0$. Let $a, b, c, d \in \mathbb{Z}$.

(1) Show that if $a \equiv c \mod N$ and $b \equiv d \mod N$, then $(a + b) \equiv (c + d) \mod N$.

(2) Show that if $a \equiv c \mod N$ and $b \equiv d \mod N$, then $(ab) \equiv (cd) \mod N$.[3]

(3) Discuss with your workmates how to use (1) and (2) to define a natural addition and multiplication on the set of congruence classes modulo N. This is delicate: we want to add/multiply two *sets* (namely, congruence classes) together to produce a third set. If you make some choices, how do you know that your operations are *well-defined*?

(4) There are exactly two congruence classes mod 2: the set of even numbers and the set of odd numbers. Make addition and multiplication tables for the operations you came up in (3) on the set $\{even, odd\}$ of all congruence classes mod 2. Is there an additive identity? Is there a multiplicative identity?

(5) Compute $([7]_5 + [-9]_5)$. Compute $[11]_3 \times [-66]_3$.

[3] Try adding and subtracting a convenient quantity from $ab - cd$.