Midterm Exam

Instructions: Solve *two* problems from Part 1 and *two* problems from Part 2. You may use any results proved in class or in the problem sets, except for the specific question being asked. You should clearly state any facts you are using. You are also allowed to use anything stated in one problem to solve a different problem, even if you have not proven it. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

Part 1: Old problems

Choose *two* of the following problems.

(1) Let V be a finite-dimensional vector space over a field F, and let $\phi : V \to V$ be a linear transformation. Show that if $\phi^2 = 0$, then $\operatorname{rank}(\phi) \leq \frac{1}{2}\dim(V)$.

(2) Let $A = \begin{bmatrix} 6 & 4 & 2 \\ 2 & 7 & 4 \end{bmatrix} \in \operatorname{Mat}_{2\times 3}(\mathbb{Z})$.

- (a) Find invertible matrices P and Q such that $B := PAQ$ is diagonal (i.e., $b_{i,j} = 0$ for all $i \neq j$).
- (b) Find all integer solutions of $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(3) Let R be a domain. We say that an R-module M is **torsionfree** if for any $r \in R$ and $m \in M$ we have $rm = 0$ implies $r = 0$ or $m = 0$.

- (a) Show that if R is a PID and M is a finitely generated torsionfree R-module, then M is free.
- (b) Give an example of a domain R and a finitely generated torisonfree R-module M that is not free.

Part 2: New problems

Choose *two* of the following problems.

(4) Let F be a field, and $A \in \operatorname{Mat}_{n\times n}(F)$. We say that A is **unipotent** if there exists some $t \in \mathbb{Z}_{\geq 1}$ such that $(A - I_n)^t = 0$.

- (a) Show that if A is unipotent, then $(A - I_n)^n = 0$.
- (b) Give a complete nonredundant set of representative for similarity classes of unipotent matrices in $\operatorname{Mat}_{3\times 3}(\mathbb{Q})$.

(5) Let R be a ring and M be a left R-module. Show that M is cyclic if and only if $M \cong R/I$ for some left ideal $I \subseteq R$.

(6) Let R be a PID. Let M be an R-submodule of R^a. Prove that M can be generated by b elements for some $b \leq a$.