Final Exam

Instructions: Solve *two* problems from Part 1 and *two* problems from Part 2. You may use any results proved in class or in the problem sets, except for the specific question being asked. You should clearly state any facts you are using. You are also allowed to use anything stated in one problem to solve a different problem, even if you have not proven it. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

Part 1: Old problems

Choose *two* of the following problems.

(1) Let A be the matrix with entries in $\mathbb{C}$

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

(a) Find the Jordan canonical form of A.

(b) Is A similar to

$$\begin{bmatrix} 0 & -9 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{bmatrix}?$$

(2) Let E be the field extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ all four complex roots of the polynomial $x^4 + 5$. That is, $E = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where

$$\alpha_1 = e^{\pi i/4}\sqrt[4]{5}, \quad \alpha_2 = e^{3\pi i/4}\sqrt[4]{5}, \quad \alpha_3 = e^{5\pi i/4}\sqrt[4]{5}, \quad \alpha_4 = e^{7\pi i/4}\sqrt[4]{5}.$$

(a) Prove that there exists[1] a field extension $\mathbb{Q} \subseteq F$ such that $F \subseteq E$, $F \subseteq \mathbb{R}$, and $[F : \mathbb{Q}] = 4$.

(b) Determine $[E : \mathbb{Q}]$ with justification.

(3) Let $F \subseteq L$ be Galois field extension of degree 45. Prove there exists a unique intermediate field E such that $[E : F] = 5$.

There are problems on the other side!

[1] Note that $\alpha_1 + \alpha_4 = \sqrt{2} \cdot \sqrt[4]{5} = \sqrt[4]{20} \in \mathbb{R}$.

Part 2: New problems

Choose *two* of the following problems.

(4) Let K/F be a finite separable field extension.

(a) Show that there exists a finite extension L/K such that L/F is Galois.

(b) Show that there exist finitely many fields E such that $F \subseteq E \subseteq K$.

(5) Let L be the splitting field over $\mathbb{Q}$ of $x^4 - 3$.

(a) Find an explicit $\mathbb{Q}$-vector space basis for L.

(b) Show that $\operatorname{Aut}(L/\mathbb{Q})$ is not abelian.

(6) Let F be a field of characteristic $p > 0$. Let $f \in F[x]$ be an irreducible polynomial of degree d, and suppose that p does not divide d. Show that f is separable.

There are problems on the other side!