Linear algebra and matrices review

(1) Let F be a field and $A \in \mathrm{Mat}_{m\times n}(F)$.
   (a) Explain why the columns of A generate $\mathrm{im}(t_A)$.
   (b) The **rank** of A is $\dim(\mathrm{im}(t_A))$. Explain why $\mathrm{rank}(A)$ is the maximal number of linearly independent columns of A.
   (c) Show that the following are equivalent:
      (i) $\mathrm{rank}(A) = m$;
      (ii) t_A is surjective;
      (iii) There is some $B \in \mathrm{Mat}_{n\times m}(F)$ such that $AB = I_m$.
   (d) Show that the following are equivalent:
      (i) $\mathrm{rank}(A) = n$;
      (ii) t_A is injective;
      (iii) There is some $B \in \mathrm{Mat}_{n\times m}(F)$ such that $BA = I_n$.
   (e) Suppose that $m = n$. List a bunch of things that are equivalent.

(2) Let R be a commutative ring and $A \in \mathrm{Mat}_{m\times n}(R)$.
   (a) If P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix,
      (i) Explain why $\ker(t_A) = \ker(t_{PA})$.
      (ii) Give a formula for $\ker(t_{AQ})$ in terms of $\ker(t_A)$ and t_Q or $t_{Q^{-1}}$.
      (iii) Explain why $\ker(t_A) \cong \ker(t_{AQ})$.
   (b) What are the analogous statements for $\mathrm{im}(t_A)$?
   (c) What do (a) and (b) say about elementary operations?
   (d) When R is a field, what does (c) say about rank?

(3) Let R be a commutative ring. Let $\mathcal{P} = \{p_1, \dots, p_m\}$ be a basis for R^m and $\mathcal{Q} = \{q_1, \dots, q_n\}$ be a basis for R^n. For $A \in \mathrm{Mat}_{m\times n}(R)$, find an explicit formula for $[t_A]^{\mathcal{P}}_{\mathcal{Q}}$ in terms of A and the matrices $P = [p_1 \cdots p_m]$ and $Q = [q_1 \cdots q_n]$.

- Let V be an F-vector space. If $I \subseteq S$ are subsets of V such that I is linearly independent and S spans V, then there is a basis B for V such that $I \subseteq B \subseteq V$.
- Let $\phi : V \to W$ be a linear transformation of F-vector spaces. Then
$$\dim(\mathrm{im}(\phi)) + \dim(\ker(\phi)) = \dim(V).$$
- For a commutative ring R and a matrix $A \in \mathrm{Mat}_{m\times n}(R)$ we have a linear transformation $t_A \colon R^n \to R^m$ by $t_A(v) = Av$.
- For a commutative ring R, an R-module homomorphism of free modules $\phi\colon V \to W$, and bases $\mathcal{B}$ for V and $\mathcal{C}$ for W, we have a matrix $[\phi]^{\mathcal{C}}_{\mathcal{B}}$ such that $[\phi]^{\mathcal{C}}_{\mathcal{B}}[v]_{\mathcal{B}} = [\phi(v)]_{\mathcal{C}}$.

(1) Let R be a ring, M an R-module, and $N \subseteq M$ be a submodule.
  (a) Show that if M can be generated by a elements, then M/N can be generated by a elements.
  (b) Show that if N can be generated by b elements and M/N can be generated by c elements, then M can be generated by $b+c$ elements.

(2) Let R be a ring and M be an R-module.
  (a) Show that M is finitely generated if and only if there is a surjective R-module homomorphism $\pi : R^m \to M$ for some m.
  (b) The set of **relations** on a (finite) set of elements $a_1, \dots, a_m \in M$ is
$$\mathrm{Rel}(a_1, \dots, a_m) = \{(r_1, \dots, r_m) \in R^m \mid r_1 a_1 + \cdots + r_m a_m = 0\}.$$
  Express $\mathrm{Rel}(a_1, \dots, a_m)$ as the kernel of a homomorphism. Deduce that the set of relations is a module.
  (c) We say that a module M is **finitely presented** if there exists a finite generating set $\{a_1, \dots, a_m\}$ for M such that $\mathrm{Rel}(a_1, \dots, a_m)$ is also finitely generated. Show that M is finitely presented if and only if there is a homomorphism of finite rank free modules $\alpha : R^n \to R^m$ such that $M \cong R^m/\mathrm{im}(\alpha)$.
  (d) Suppose that R is commutative. Show that M is finitely presented if and only if there is some matrix A such that $M \cong R^m/\mathrm{im}(t_A)$.

(3) Let R be a commutative ring, and $D \in \mathrm{Mat}_{m\times n}(R)$ be a diagonal matrix (meaning $d_{ij} = 0$ for $i \neq j$) with nonzero diagonal entries $d_{11}, \dots, d_{rr}$. Prove that the module presented by D is isomorphic to
$$R/(d_{11}) \oplus R/(d_{22}) \oplus \cdots \oplus R/(d_{rr}) \oplus R^{m-r}.$$

- Let R be a ring, M a module, and $S \subseteq M$. Then S **generates** M if no proper submodule of M contains S. Equivalently, every element of M is an R-linear combination of elements of S.
- Let R be a commutative ring and $A \in \mathrm{Mat}_{m\times n}(R)$. The **module presented by** A is $R^m/\mathrm{im}(t_A)$.