

THE FUNDAMENTAL THEOREM OF GALOIS THEORY

THE FUNDAMENTAL THEOREM OF GALOIS THEORY: Let L/F be a finite Galois extension. There is a bijection

$$\begin{array}{ccc} \{\text{fields } E \mid F \subseteq E \subseteq L\} & \longleftrightarrow & \{\text{subgroups } H \leq \text{Gal}(L/F)\} \\ E & \mapsto & \text{Gal}(L/E) \\ L^H & \longleftarrow & H \end{array}$$

Moreover, this bijective correspondence enjoys the following properties:

- (1) $\xrightarrow{\text{Gal}(L/\star)}$ and $\xleftarrow{L^\star}$ each reverse the order of inclusion:
 $E \subseteq E' \iff \text{Gal}(L/E') \leq \text{Gal}(L/E)$; equivalently
 $L^H \subseteq L^{H'} \iff H' \leq H$.
- (2) $\xrightarrow{\text{Gal}(L/\star)}$ and $\xleftarrow{L^\star}$ convert between degrees of extensions and indices of subgroups:
 $[E : F] = [\text{Gal}(L/F) : \text{Gal}(L/E)]$; equivalently
 $[\text{Gal}(L/F) : H] = [L^H : F]$.
- (3) Normal subgroups correspond to intermediate fields that are Galois over F :
 E/F is Galois if and only if $\text{Gal}(L/E) \trianglelefteq \text{Gal}(L/F)$; equivalently
 $N \trianglelefteq G$ if and only if L^N/F is Galois.
 In this case, $\text{Gal}(E/F) = \text{Gal}(L/F)/\text{Gal}(L/E)$.

(1) Let L be the splitting field of $x^3 - 2$ over \mathbb{Q} . From earlier work, we know that:

- $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$.
- $\text{Gal}(L/\mathbb{Q}) = \langle \sigma, \tau \rangle$, with $\sigma \begin{bmatrix} \sqrt[3]{2} \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \sqrt[3]{2} \\ \omega \end{bmatrix}$ and $\tau \begin{bmatrix} \sqrt[3]{2} \\ \omega \end{bmatrix} = \begin{bmatrix} \sqrt[3]{2} \\ \omega^2 \end{bmatrix}$.
- $\text{Gal}(L/\mathbb{Q}) \cong S_3$ via $\sigma \mapsto (1\ 2\ 3)$ and $\tau \mapsto (2\ 3)$.

- (a)** List all subgroups of S_3 , and draw a diagram indicating which is contained in which.
- (b)** Use the previous part and the Fundamental Theorem to determine how many¹ intermediate fields E are between \mathbb{Q} and L . Then draw a diagram to indicate containments among the fields.
- (c)** Compute the index of each subgroup of S_3 . Use this and the Fundamental Theorem to determine $[E : \mathbb{Q}]$ for each intermediate field E .
- (d)** Determine which subgroups of S_3 are normal. Use this and the Fundamental Theorem to determine which intermediate fields are Galois over \mathbb{Q} .
- (e)** Explicitly identify the intermediate fields E .

(2) Let L/F be a finite Galois extension.

- (a)** When is every intermediate field $E \subseteq L$ Galois over F ?
- (b)** If $[L : F] = n$ and $n = p^e m$ for some prime p and positive integer m that is not a multiple of p , explain why there exists an intermediate field E with $[E : F] = m$.

(3) Proof² of Fundamental Theorem:

- (a)** Show that the maps $\xrightarrow{\text{Gal}(L/\star)}$ and $\xleftarrow{L^\star}$ give the bijection as stated in the Theorem.
- (b)** Prove part (1) of The Fundamental Theorem of Galois Theory.
- (c)** Prove part (2) of The Fundamental Theorem of Galois Theory.

¹Do *not* explicitly identify the fields yet; just make up a placeholder name for each intermediate field E .

²Hint: Use Artin's Theorem and its Corollaries generously!