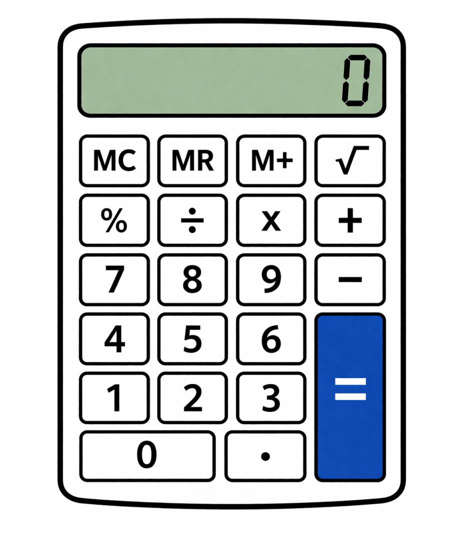
DEFINITION: A **dollar store calculator** is a device that allows you to enter numbers with finite decimal expansion, and apply the operations $+, -, \times, \div$ and $\sqrt{}$. A dollar store calculator can only work with real numbers.



(a) What is the set of numbers that can be computed exactly in a dollar store calculator *without using* the $\sqrt{}$ key?

$\mathbb{Q}$

(b) Explain how $\sqrt[4]{2}$ can be computed exactly with a dollar store calculator.

$\sqrt{\sqrt{2}}$

(c) Carefully explain[1] why neither $\sqrt[3]{2}$ nor $\sqrt[4]{2}$ can be computed exactly with a dollar store calculator using the $\sqrt{}$ key *only once* (at any point, not necessarily at the end).

Suppose that β can be obtained using $\sqrt{}$ only once. Let $\alpha \in \mathbb{Q}$ be the number that you apply $\sqrt{}$ to. Since any operation after that is essentially $+, -, \times, \div$ with rational numbers, $\beta \in \mathbb{Q}(\sqrt{\alpha})$. Note that $[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}] = 2$, and $\beta \in \mathbb{Q}(\sqrt{\alpha})$, we must have $[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}(\beta)][\mathbb{Q}(\beta) : \mathbb{Q}]$ by the Degree Formula. Thus $[\mathbb{Q}(\beta) : \mathbb{Q}] \mid 2$. However, the minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$ since this is irreducible (by Gauss+Eisenstein) and monic, and has this as a root; thus $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. This is a contradiction. Similarly for $\sqrt[4]{2}$.

[1]Hint: Let α be the number that $\sqrt{}$ is applied to. Show that the final result must be in $\mathbb{Q}(\sqrt{\alpha})$, and compare $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}]$ with $[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}]$.

(d) Carefully explain[2] why π *cannot* be computed exactly with a dollar store calculator. You can use any facts from analysis that we have assumed in this class.

Suppose that π can be obtained, and let $\alpha_1, \ldots, \alpha_n$ be as in the hint. Note that $\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{i+1}})$ is finite, and hence algebraic over $\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_i})$, since $\sqrt{\alpha_{i+1}}$ is a root of the polynomial $x^2 - \alpha_{i+1} \in \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_i})$ It follows that $\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$ is algebraic over $\mathbb{Q}$. Since the final result π is in this field, then π must be algebraic over $\mathbb{Q}$. This contradicts a fact from analysis.

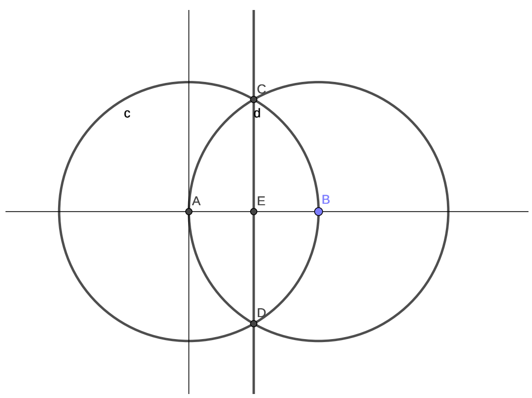
(e) Carefully explain[2] why $\sqrt[3]{2}$ *cannot* be computed exactly with a dollar store calculator.

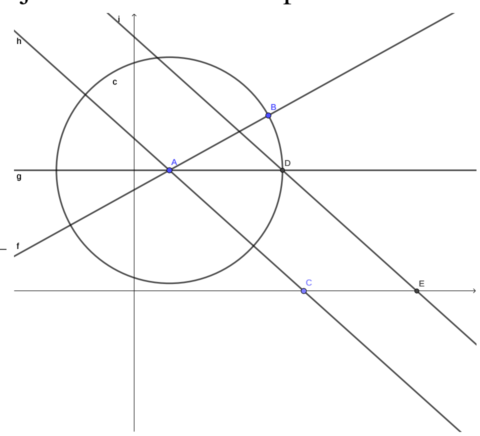
Suppose that $\sqrt[3]{2}$ can be obtained, and let $\alpha_1, \ldots, \alpha_n$ be as in the hint. The previous argument shows that $[\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{i+1}}) : \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_i})]$ is at most 2; in particular, it is either 1 or 2 for each i. By the Degree Formula, $[\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) : \mathbb{Q}] = 2^k$ for some $k \leq n$. By the argument from (c), we must have $3 \mid 2^k$; this is a contradiction.

[2]Hint: Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the numbers that $\sqrt{\ }$ is applied to. What can you say about $\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$?

A **classical compass and straightedge construction** is a rule to create from an input finite subset of *marked points* $S \subseteq \mathbb{R}^2$ another point $Z \in \mathbb{R}^2$ by a sequence of the following rules:

- Take any point $Z \in S$.
- If L is the line between two points $P, Q \in S$, and L' is the line between two points $P', Q' \in S$, then adjoin the intersection point $L \cap L'$ to S.
- If L is the line between two points $P, Q \in S$, and C is the circle centered at $P' \in S$ passing through $Q' \in S$, then adjoin the intersection points $L \cap C$ to S.
- If C is the circle centered at $P \in S$ passing through $Q \in S$, and C' is the circle centered at $P' \in S$ passing through $Q' \in S$, then adjoin the intersection points $C \cap C'$ to S.





LEMMA: Given points P, Q, P', Q', the coordinates of a point X in $L \cap L'$, $L \cap C$, or $C \cap C'$ as in the constructions above can be computed from the coordinates of P, Q, P', Q' by a sequence of the operations $+, -, \times, \div, \sqrt{}$.

(f) Suppose that $S = \{(0,0), (1,0)\}$. Use the LEMMA to explain why the coordinates of any point that can be obtained from S by a classical compass and straightedge construction can also be obtained from a dollar store calculator.

(g) SQUARING THE CIRCLE: Given two points P, Q, show that is impossible to use a classical compass and straightedge construction to give a line segment $\overline{RS}$ such that the area of the square with base $\overline{RS}$ equals the area of the circle with center P passing through Q.

(h) DOUBLING THE CUBE: Given two points P, Q, show that is impossible to use a classical compass and straightedge construction to give a line segment $\overline{RS}$ such that the volume of the cube with edge $\overline{RS}$ equals twice the volume of the cube with edge $\overline{PQ}$.