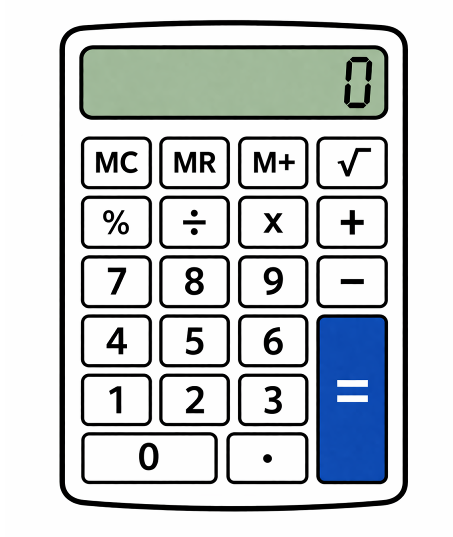
DEFINITION: A **dollar store calculator** is a device that allows you to enter numbers with finite decimal expansion, and apply the operations $+, -, \times, \div$ and $\sqrt{\ }$. A dollar store calculator can only work with real numbers.



(a) What is the set of numbers that can be computed exactly in a dollar store calculator *without using* the $\sqrt{\ }$ key?

(b) Explain how $\sqrt[4]{2}$ can be computed exactly with a dollar store calculator.

(c) Carefully explain[1] why neither $\sqrt[3]{2}$ nor $\sqrt[4]{2}$ can be computed exactly with a dollar store calculator using the $\sqrt{\ }$ key *only once* (at any point, not necessarily at the end).

(d) Carefully explain[2] why π *cannot* be computed exactly with a dollar store calculator. You can use any facts from analysis that we have assumed in this class.

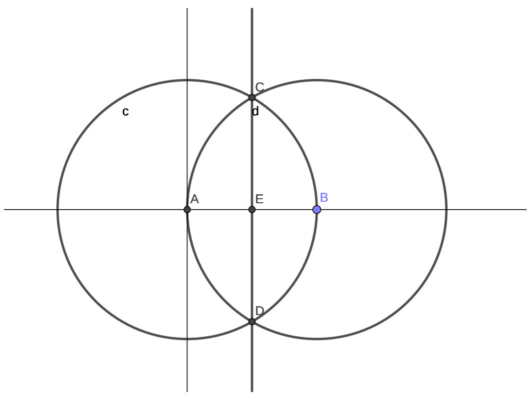
(e) Carefully explain[2] why $\sqrt[3]{2}$ *cannot* be computed exactly with a dollar store calculator.

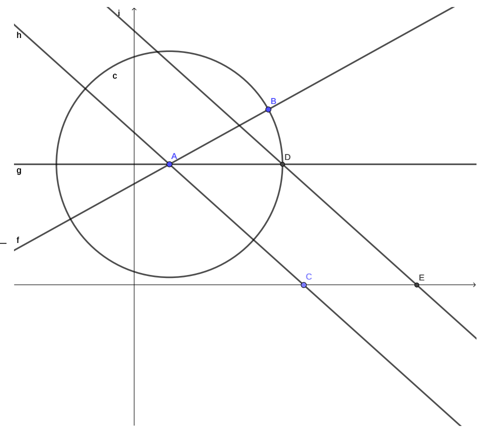
[1] Hint: Let α be the number that $\sqrt{\ }$ is applied to. Show that the final result must be in $\mathbb{Q}(\sqrt{\alpha})$, and compare $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}]$ with $[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}]$.

[2] Hint: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the numbers that $\sqrt{\ }$ is applied to, in order. What can you say about $\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$?

A **classical compass and straightedge construction** is a rule to create from an input finite subset of *marked points* $S \subseteq \mathbb{R}^2$ another point $Z \in \mathbb{R}^2$ by a sequence of the following rules:

- Take any point $Z \in S$.
- If L is the line between two points $P, Q \in S$, and L' is the line between two points $P', Q' \in S$, then adjoin the intersection point $L \cap L'$ to S.
- If L is the line between two points $P, Q \in S$, and C is the circle centered at $P' \in S$ passing through $Q' \in S$, then adjoin the intersection points $L \cap C$ to S.
- If C is the circle centered at $P \in S$ passing through $Q \in S$, and C' is the circle centered at $P' \in S$ passing through $Q' \in S$, then adjoin the intersection points $C \cap C'$ to S.





LEMMA: Given points P, Q, P', Q', the coordinates of a point X in $L \cap L'$, $L \cap C$, or $C \cap C'$ as in the constructions above can be computed from the coordinates of P, Q, P', Q' by a sequence of the operations $+, -, \times, \div, \sqrt{}$.

(f) Suppose that $S = \{(0,0), (1,0)\}$. Use the LEMMA to explain why the coordinates of any point that can be obtained from S by a classical compass and straightedge construction can also be obtained from a dollar store calculator.

(g) SQUARING THE CIRCLE: Given two points P, Q, show that is impossible to use a classical compass and straightedge construction to give a line segment $\overline{RS}$ such that the area of the square with base $\overline{RS}$ equals the area of the circle with center P passing through Q.

(h) DOUBLING THE CUBE: Given two points P, Q, show that is impossible to use a classical compass and straightedge construction to give a line segment $\overline{RS}$ such that the volume of the cube with edge $\overline{RS}$ equals twice the volume of the cube with edge $\overline{PQ}$.