

THE DEGREE FORMULA AND ALGEBRAIC EXTENSIONS

DEGREE FORMULA: Let $F \subseteq L \subseteq K$ be field extensions. Then $[K : F] = [K : L] \cdot [L : F]$. In particular, K/F is finite if and only if K/L and L/F are both finite.

DEFINITION: A field extension L/F is **algebraic** if every $\alpha \in L$ is algebraic over F .

PROPOSITION: Any finite field extension is algebraic.

THEOREM: Let $F \subseteq L \subseteq K$ be field extensions. Then K/F is algebraic if and only if K/L and L/F are both algebraic.

- (1)** Using the Degree Formula: Consider the field $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
- (a)** Let $E = \mathbb{Q}(\sqrt{2})$. Use the polynomials $x^2 - 2 \in \mathbb{Q}[x]$ and $x^2 - 3 \in E[x]$ to quickly show that $[E : \mathbb{Q}] \leq 2$ and $[F : E] \leq 2$.
 - (b)** Explain why $\mathbb{Q} \subsetneq E \subsetneq F$, and deduce that $[E : \mathbb{Q}] = 2$ and $[F : E] = 2$.
 - (c)** Show that $[F : \mathbb{Q}] = 4$.
 - (d)** We showed earlier that $F = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ and that $\sqrt{2} + \sqrt{3}$ is a root of the polynomial $p(x) = x^4 - 10x^2 + 1$. Deduce from these facts and from (c) that $p(x)$ is irreducible over $\mathbb{Q}[x]$.
- (2)** Use¹ the Degree Formula to prove the Proposition.
- (3)** Proof of Theorem (using the Degree Formula):
- (a)** Prove the (\implies) direction using the definitions.
 - (b)** For the (\impliedby) direction, take $\alpha \in K$. Explain why there exist $a_0, a_1, \dots, a_n \in L$ such that α is algebraic over $F(a_0, \dots, a_n)$.
 - (c)** Consider the tower of field extensions

$$F \subseteq F(a_0) \subseteq F(a_0, a_1) \subseteq \dots \subseteq F(a_0, \dots, a_n) \subseteq F(a_0, \dots, a_n, \alpha).$$
 Explain why each inclusion is finite. Then explain why $F(a_0, \dots, a_n, \alpha)/F$ is finite.
 - (d)** Deduce the Theorem.
- (4)** Prove the Degree Formula.

¹Hint: If L/F is finite and $\alpha \in L$, consider $F \subseteq F(\alpha) \subseteq L$.

THEOREM (SIMPLE EXTENSIONS): Let L/F be a field extension and $\alpha \in L$.

- (1) $I := \{f(x) \in F[x] \mid f(\alpha) = 0\}$ is an ideal in $F[x]$.
- (2) $I \neq 0 \iff \alpha$ is algebraic over F .
- (3) If α is algebraic over F , then the unique monic generator of I , denoted $m_{\alpha,F}(x)$, is irreducible.
- (4) If α is algebraic over F , then $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$.
- (5) α is algebraic over $F \iff [F(\alpha) : F] < \infty$.
In this case, $[F(\alpha) : F] = \deg(m_{\alpha,F})$.
- (6) α is transcendental over $F \iff [F(\alpha) : F] = \infty$.
In this case, $F(\alpha) \cong F(x)$, the field of rational functions in one variable over F .